Translation-invariance of two-dimensional Gibbsian systems of particles with internal degrees of freedom

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Abstract

One of the main objectives of equilibrium state statistical physics is to analyze which symmetries of an interacting particle system in equilibrium are broken or conserved. Here we present a general result on the conservation of translational symmetry for two-dimensional Gibbsian particle systems. The result applies to particles with internal degrees of freedom and fairly arbitrary interaction, including the interesting cases of discontinuous, singular, and hard core interaction. In particular we thus show the conservation of translational symmetry for the continuum Widom Rowlinson model and a class of continuum Potts type models.

Key words: Gibbs measures, Mermin-Wagner theorem, translation, hard core, singularity, Widom Rowlinson model, Potts model, percolation.

1 Introduction

It is well known that probability theory provides a mathematically rigorous setting to investigate problems from equilibrium state statistical physics. Here the object of consideration is a system of interacting particles, where the number of particles is huge and thus assumed to be infinite. Such a particle system is given by specifying restrictions on particle positions (lattice setting versus point particle setting), the internal properties of the particles (such as magnetic spin, electric charge or particle type), and the interaction between particles. The equilibrium states of a specific particle system are then modeled by Gibbsian processes, as introduced by R. L. Dobrushin (see [D1] and [D2]), O. E. Lanford and D. Ruelle (see [LR]). The main objective usually is to find out whether the system exhibits a phase transition, i.e. whether there is more than one equilibrium state. In order to study this problem, the crucial task is to investigate which of the system's symmetries are broken and which are conserved, and it would be desirable to have general results stating under which conditions certain symmetries are conserved. While in more than two spatial dimensions such general results can not be expected to hold (as here all symmetries are believed to be broken easily), and in one dimension the situation is almost trivial (as under very weak assumptions all symmetries are conserved), the case of two dimensions is interesting. Here it is useful to distinguish between discrete and continuous symmetries and also between internal symmetries (i.e. symmetry transformations concerning the inner properties of particles) and spatial symmetries (such as translation and rotation). In order to investigate the behavior of a particle system under

translations and rotations it is natural to consider a point particle setting, as in a lattice setting all spatial symmetries are bound to be discrete. In the following the attention is thus restricted to interacting particle systems in a point particle setting in two dimensions.

The current knowledge about such systems is the following: It is believed that discrete internal symmetries in general may be broken, but so far this has been shown only for very few systems, e.g. the Widom Rowlinson model considered by D. Ruelle [Ru2] or the continuum Potts model considered by H.-O. Georgii and O. Häggström [GH]. In contrast, continuous internal symmetries are conserved under weak assumptions on the interaction. The first result in this direction was obtained by S. Shlosman [S], building on earlier ideas of M. Mermin and H. Wagner [MW]. We gave a more general version of this result in [Ri1], which includes the case of discontinuous interaction, using ideas of D. Ioffe, S. Shlosman and Y. Velenik [ISV]. While it is expected that rotational symmetry may be broken, so far this could not be established for any realistic particle system, but this conjecture is supported by recent work of F. Merkl and S. Rolles [MR], for example. Translational symmetry is conserved under weak assumptions on the interaction. This was first shown by E. Fröhlich and C.-E. Pfister [FP1] and [FP2], and we obtained a more general result [Ri2], which for example includes the interesting case of the hard disc model. The last two results both concern particles without internal degrees of freedom. However, many interesting models of statistical physics, such as the Widom Rowlinson or the Potts model, feature particles with spins. Here we will show, how to overcome conceptual and technical difficulties that arise due to the incorporation of spins, and thus we obtain a fairly general result on the conservation of translational symmetry for particles with any internal degrees of freedom and for interactions that are allowed to have discontinuities, singularities, or hard cores. This establishes the conservation of translational symmetry for the continuum Widom Rowlinson model and a large class of continuum Potts type models, for example. While parts of the proof of the main theorem will be similar to the corresponding parts in [Ri2], we decided to repeat these arguments for the convenience of the reader, so that the article is self-contained.

We start Section 2 by giving an equivalent condition for a measure to be invariant under a transformation (Lemma 1), which will be useful for establishing the conservation of symmetries. We next confine ourselves to the special case of translational symmetry, considering a class of Potts type potentials. The corresponding result (Theorem 1), which is of interest on its own, will follow from the general case presented afterwards. For this general case we define a class of potentials (Definition 1) for that translational symmetry is conserved (Theorem 2). After a few comments on some aspects of this class concerning hard cores (Lemmas 2 and 3) we give sufficient conditions for potentials to belong to this class (Lemmas 4 and 5). The precise setting is given in Section 3, and the proofs of the lemmas from Sections 2 and 3 are relegated to Section 4. In Section 5 we will give the proof of Theorem 2. The proofs of the corresponding lemmas are relegated to Section 6.

2 Results

2.1 Conservation of symmetries

We consider particles in the plane $\mathbb{R}^2$. Every particle is allowed to have internal degrees of freedom, encoded in the so called spin of the particle. The spin is assumed to be an element of some measurable spin space (or mark space) $(S, \mathcal{F}_S)$, on which a probability measure λ_S is given as a reference measure. We require the diagonal in $S \times S$ to be measurable w.r.t $\mathcal{F}_S \otimes \mathcal{F}_S$, but we will not assume any topological properties of S. The particle space will be abbreviated by $\mathbb{R}^2_S := \mathbb{R}^2 \times S$. We fix a chemical potential $-\log z$, where $z > 0$ is a given activity parameter. The particles may interact via a *pair potential* U modelled by a measurable function

$$U : (\mathbb{R}^2_S)^2 \;\to\; \overline{\mathbb{R}} \;:=\; \mathbb{R} \cup \{\infty\}$$

that is symmetric in that $U(y_1, y_2) = U(y_2, y_1)$. The set of equilibrium states corresponding to a particular choice of U and z can be modelled by the set of Gibbsian point processes, which are defined to be certain probability measures on the space $(\mathcal{Y}, \mathcal{F}_\mathcal{Y})$ of all particle configurations, see Section 3.3. A bimeasurable transformation $\tau : \mathbb{R}^2_S \to \mathbb{R}^2_S$ is called a *symmetry* of U if U is τ-invariant, i.e.

$$U(\tau(y_1), \tau(y_2)) = U(y_1, y_2) \quad \text{ for all } y_1, y_2 \in \mathbb{R}^2_S.$$

Such a transformation τ also defines a transformation on the configuration space $\mathcal{Y}$, where every single particle of a given configuration is transformed by τ, and an equilibrium state μ is said to be τ-invariant if $\mu \circ \tau^{-1} = \mu$. It is natural to ask, whether the equilibrium states of a particle system corresponding to U and z are invariant under a given symmetry of U. If this is indeed the case, the symmetry is said to be conserved, otherwise it is said to be broken.

There are several strategies to establish the conservation of symmetries. One is to use the concept of relative entropy and to exploit certain entropy estimates, see Section 2.3.3. of [ISV]. Another one builds on a certain inequality for Gibbsian specifications, see Proposition (9.1) of [G]. The latter approach uses the convexity of the set of Gibbs measures, tail triviality of extremal Gibbs measures and extreme decomposition, thus requiring the spin space to be standard Borel. In the following we present a variant of this approach, which works in a general setting and admits a straightforward proof via convexity.

Lemma 1 *Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, $\mathcal{A} \subset \mathcal{F}$ an algebra on Ω such that $\sigma(\mathcal{A}) = \mathcal{F}$, and τ a transformation on Ω, i.e. $\tau : \Omega \to \Omega$ a bimeasurable mapping. μ is τ-invariant if and only if the following condition holds:*

$$\forall A \in \mathcal{A} : \quad \mu(\tau A) \,+\, \mu(\tau^{-1} A) \,\geq\, 2\mu(A). \tag{2.1}$$

The proof will be given in Section 4. For a more detailed account on how to use this lemma in order to show the conservation of translational symmetry, see Subsection 3.5. From now on we will restrict our attention to spatial translations of particles.

2.2 Widom Rowlinson and Potts type potentials

As the definition of the class of potentials for that we will show the conservation of translational symmetry is fairly general, but also fairly complicated, we first would like to present the result for a certain class of Potts type potentials. This class includes finite state Widom Rowlinson potentials as wells as step potentials, as considered by J. L. Lebowitz and E. H. Lieb in [LL] as a continuum analogue of the Potts model. For a given finite spin space S (describing different types of particles) we define a *Potts type potential* to be of the form

$$U(x_1, \sigma_1, x_2, \sigma_2) := \phi_{\sigma_1,\sigma_2}(|x_1 - x_2|_h),$$

where $|.|_h$ is a norm on $\mathbb{R}^2$ and $(\phi_{\sigma_1\sigma_2})_{\sigma_1,\sigma_2 \in S}$ is a family of interactions, i.e. $\phi_{\sigma_1\sigma_2} : \mathbb{R}_+ := [0, \infty[\to \overline{\mathbb{R}}$ is measurable and we have $\phi_{\sigma_1\sigma_2} = \phi_{\sigma_2\sigma_1}$ for all σ_1, σ_2.

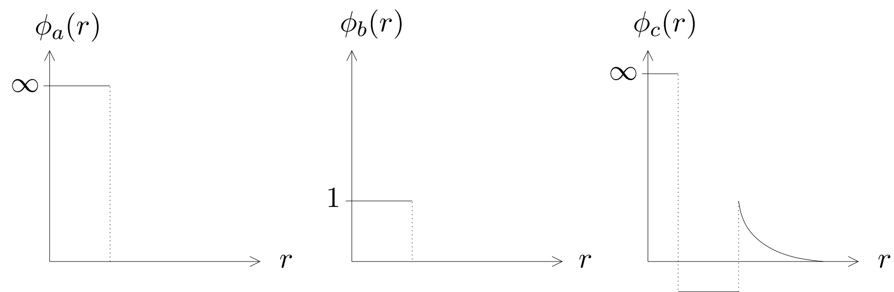

Figure 1: Some examples of well behaved functions

We call a function $\phi : \mathbb{R}_+ \to \overline{\mathbb{R}}$ *well behaved* if there are $0 \leq r_0 < \ldots < r_n$ $(n \geq 0)$ such that $\phi(r) = \infty$ for $r < r_0$, $\phi(r) = 0$ for $r > r_n$, ϕ is continuous on every interval $]r_i, r_{i+1}[$ and in every point $r_1, \ldots, r_n$ the left and right limit exist. Figure 1 shows some examples of well behaved functions. Functions of type ϕ_a, ϕ_b and ϕ_c are used in the definition of a Widom Rowlinson potential, a continuum Potts potential and a slightly more complicated Potts type potential respectively. All spatial translations of particles are symmetries of Potts type potentials, and the following theorem states the conservation of these symmetries.

Theorem 1 *Let S be a finite spin space endowed with the equidistribution as reference measure and $z > 0$ be an activity parameter. Let U be a Potts type potential corresponding to a norm $|.|_h$ on $\mathbb{R}^2$ and a family of interactions $(\phi_{\sigma_1\sigma_2})_{\sigma_1,\sigma_2 \in S}$. If all the functions $\phi_{\sigma_1\sigma_2}$ are nonnegative and well behaved, then every Gibbs measure corresponding to U and z is translation-invariant.*

We note that nonnegativity of the potential is assumed only in order to avoid introducing superstability at this point. In Section 2.5, Theorem 1 will be deduced from the general case (Theorem 2) presented below.

2.3 General case

In this general case we consider translations in a fixed direction $\vec{a}$ ($\vec{a} \in \mathbb{R}^2$ with $|\vec{a}|_2 = 1$). The corresponding group of translation transformations is defined by

$$g_t : \mathbb{R}^2_S \to \mathbb{R}^2_S, \quad g_t(x,\sigma) := (x,\sigma) + \vec{a}t := (x + \vec{a}t, \sigma) \quad (t \in \mathbb{R}).$$

We call a potential U (or a Gibbsian point process μ) invariant under translations in direction $\vec{a}$ or simply $\vec{a}$-invariant if U (or μ respectively) is invariant under g_t for all $t \in \mathbb{R}$. Translation-invariance is equivalent to $\vec{a}$-invariance in every direction $\vec{a}$. As there might be interesting potentials that are $\vec{a}$-invariant for some direction $\vec{a}$, but not for every direction, we investigate the conservation of $\vec{a}$-translational symmetry rather than translational symmetry.

In order to describe a class of potentials for that $\vec{a}$-symmetry is conserved, we now define some important properties of sets, functions, and potentials. We call a function $f : (\mathbb{R}^2_S)^2 \to \overline{\mathbb{R}}$

$\vec{a}$-invariant	if	$f(y + t\vec{a}, y' + t\vec{a}) = f(y, y') \ \forall\, y, y' \in \mathbb{R}^2_S, t \in \mathbb{R}$,
symmetric	if	$f(y, y') = f(y', y) \ \forall\, y, y' \in \mathbb{R}^2_S$ and
of *bounded range*	if	$\{\|y - y'\| : f(y, y') \neq 0\}$ is bounded.

Here the distance of two particles is defined to be the distance of the positions of the particles. The above definition of course does not depend on the choice of norm $|.|$, but for sake of definiteness let $|.|$ be the maximum norm on $\mathbb{R}^2$. We say that a set $A \subset (\mathbb{R}^2_S)^2$ is $\vec{a}$-invariant, symmetric, or of bounded range if the corresponding indicator function 1_A has this property. We call A a *standard set* if it is measurable, symmetric, and of bounded range. Let us call U a *standard potential* if it is measurable, symmetric, and its hard core

$$K^U := \{U = +\infty\}$$

is a standard set, i.e. if its hard core is of bounded range. Usually the hard core can be described in terms of a norm, which is the case for Potts type potentials as described above, for example, but in our setup we are able to treat fairly general hard cores. We also need regularity properties. We call a potential U *$\vec{a}$-continuous*, *$\vec{a}$-equicontinuous*, or *$\vec{a}$-smooth* on a set A if the family of functions

$$\varphi^U_{y_1,y_2}(t) : \mathbb{R} \to \overline{\mathbb{R}}, \quad t \mapsto U(y_1, y_2 + t\vec{a}) \quad ((y_1, y_2) \in A)$$

is continuous, equicontinuous, or smooth in $t = 0$. In the case of smoothness we define the $\vec{a}$-derivatives of U in A by

$$\partial^2_{\vec{a}} U(y_1, y_2) := \frac{d^2}{dt^2} \varphi^U_{y_1,y_2}(0),$$

and for a given function $\psi : (\mathbb{R}^2_S)^2 \to \mathbb{R}_+$ we say that the $\vec{a}$-derivatives of U are *dominated* by ψ on A if

$$\partial^2_{\vec{a}} U(y_1, y_2 + t\vec{a}) \le \psi(y_1, y_2) \ \text{ for all } (y_1, y_2) \in A, t \in [-1, 1] \text{ s.t. } (y_1, y_2 + t\vec{a}) \in A.$$

In the context of ψ-domination we will use the notion of a *bounded partially square integrable function (bpsi-function)*, which is defined to be a measurable, symmetric function $\psi : (\mathbb{R}^2_S)^2 \to \mathbb{R}_+$ satisfying

$$\|\psi\| \, < \, \infty \quad \text{and} \quad \sup_{y_1 \in \mathbb{R}^2_S} \int \psi(y_1, y_2)|y_1 - y_2|^2 dy_2 \, < \, \infty,$$

where $\|.\|$ is the supremum norm of a function. In order to be able to control the potential in a neighborhood of a given set, we introduce the notion of the *ϵ-$\vec{a}$-enlargement* $K_{\epsilon,\vec{a}}$ of a set $K \subset (\mathbb{R}^2_S)^2$ for a given $\epsilon > 0$, defined by

$$K_{\epsilon,\vec{a}} \, := \, \{(y_1, y_2 + r\vec{a}) : (y_1, y_2) \in K, -\epsilon < r < \epsilon\}.$$

We note that the ϵ-$\vec{a}$-enlargement of an $\vec{a}$-invariant standard set again is an $\vec{a}$-invariant, symmetric set of bounded range. However, it is not necessarily measurable, so we need to be a bit more careful. Given an $\vec{a}$-invariant standard set K we define K', K'' to be *measurable $\vec{a}$-enlargements of* K if for some $\epsilon > 0$

$$K' \text{ and } K'' \text{ are standard sets}, \quad K_{\epsilon,\vec{a}} \subset K' \quad \text{and} \quad (K')_{\epsilon,\vec{a}} \subset K''.$$

If U is a potential, $z > 0$ is an activity parameter and $\mathcal{Y}_0$ is a set of boundary conditions, we say that the triple $(U, z, \mathcal{Y}_0)$ is admissible if all conditional Gibbs distributions corresponding to U and z with boundary condition taken from $\mathcal{Y}_0$ are well defined, see Definition 2 in Section 3.3. Important examples are the cases of superstable potentials with tempered boundary configurations and nonnegative potentials with arbitrary boundary conditions, see Section 3.4. For admissible $(U, z, \mathcal{Y}_0)$ the set of Gibbs measures $\mathcal{G}_{\mathcal{Y}_0}(U, z)$ corresponding to U and z with full weight on configurations in $\mathcal{Y}_0$ is a well defined object. Finally we need bounded correlations: For admissible $(U, z, \mathcal{Y}_0)$ we call $\xi \in \mathbb{R}$ a Ruelle bound if the correlation function of every Gibbs measure $\mu \in \mathcal{G}_{\mathcal{Y}_0}(U, z)$ is bounded by powers of ξ in the sense of (3.3) in Section 3.3.

Definition 1 *Let $(U, z, \mathcal{Y}_0)$ be admissible with Ruelle bound ξ, where $U : (\mathbb{R}^2_S)^2 \to \overline{\mathbb{R}}$ is an $\vec{a}$-invariant standard potential. We say that U is $\vec{a}$-smoothly approximable if there is a decomposition of U into a smooth part $\bar{U}$ and a small part u in the following sense: We have an $\vec{a}$-invariant standard set $K \supset K^U$ and measurable symmetric $\vec{a}$-invariant functions $\bar{U}, u : K^c \to \mathbb{R}$ such that $U = \bar{U} - u$, $u \geq 0$, $\bar{U}$ has ψ-dominated $\vec{a}$-derivatives on K^c for some bpsi-function ψ, and*

$$\begin{aligned}
&\sup_{y_1 \in \mathbb{R}^2_S} \int (1_{K^c}\tilde{u})(y_1, y_2)|y_1 - y_2|^2 \, dy_2 \, < \, \infty \;\; \textit{and} \\
&\sup_{y_1 \in \mathbb{R}^2_S} \int (1_{K'' \setminus K^U} + 1_{K^c}\tilde{u})(y_1, y_2) \, dy_2 \, < \, \frac{1}{z\xi}
\end{aligned} \tag{2.2}$$

for some measurable $\vec{a}$-enlargements K', K'' of K and $\tilde{u} := 1 - e^{-u} \leq u \wedge 1$.

The class of smoothly approximable standard potentials is a rich class of potentials. An $\vec{a}$-smoothly approximable $\vec{a}$-invariant standard potential may have a singularity or a hard core at the origin, and the type of convergence into the singularity or the hard core is fairly arbitrary, as we have not imposed any condition on U in $K \setminus K^U$. For small activity z the last condition of (2.2) holds for large sets K'', so K can be chosen to be a large set, which relaxes the conditions on U. The small part u of U is not assumed to satisfy any regularity conditions, so that U doesn't have to be smooth or continuous. We note that Definition 1 does not depend on the choice of the norm $|.|$.

The above definition may seem overly complicated. Nevertheless we present it in the given form in order to include as many potentials as possible in the class of $\vec{a}$-smoothly approximable potentials. For some comments on Definition 1 and simplifications in several special cases we refer to the following subsections. Beforehand however, we would like to present our main result:

Theorem 2 *Let $(U, z, \mathcal{Y}_0)$ be admissible with Ruelle bound, where $U : (\mathbb{R}^2_S)^2 \to \overline{\mathbb{R}}$ is a standard potential. If U is $\vec{a}$-invariant and $\vec{a}$-smoothly approximable, then every Gibbs measure $\mu \in \mathcal{G}_{\mathcal{Y}_0}(U, z)$ is $\vec{a}$-invariant.*

2.4 Measurable enlargements

In Definition 1 the hard core K^U may be of fairly arbitrary size and shape. The only condition on K^U is the existence of an $\vec{a}$-invariant standard set $K \supset K^U$ and of measurable $\vec{a}$-enlargements K', K'' of K such that $K'' \setminus K$ and $K \setminus K^U$ are not too big in the sense of the second inequality of (2.2). In the following we present possible constructions of measurable $\vec{a}$-enlargements K', K'' for a given $\vec{a}$-invariant standard set K.

(a) Even if the ϵ-$\vec{a}$-enlargements of a measurable set K in general need not be measurable again, they often are. If so, we simply set $K' = K_{\epsilon,\vec{a}}$ and $K'' = (K')_{\epsilon,\vec{a}} = K_{2\epsilon,\vec{a}}$ to construct measurable enlargements of K.

Lemma 2 *Let $A \subset (\mathbb{R}^2_S)^2$ be a measurable set with all $\vec{a}$-cross sections $A(y_1, y_2, \vec{a}) := \{r \in \mathbb{R} : (y_1, y_2 + r\vec{a}) \in A\}$ $(y_1, y_2 \in \mathbb{R}^2_S)$ satisfying*

$$\forall\, U \subset \mathbb{R} \text{ open}: \quad U \cap A(y_1, y_2, \vec{a}) \neq \emptyset \Rightarrow \lambda^2(U \cap A(y_1, y_2, \vec{a})) > 0. \qquad (2.3)$$

Then every ϵ-$\vec{a}$-enlargement $A_{\epsilon,\vec{a}}$ $(\epsilon > 0)$ is measurable again. For example (2.3) is satisfied if the set of interior points of $A(y_1, y_2, \vec{a})$ is dense in $A(y_1, y_2, \vec{a})$.

Condition (2.3) concerns only the topological structure of the $\vec{a}$-cross sections of A, is easy to be verified, and holds in the case that each $\vec{a}$-cross section of A is open, for example.

(b) Often we may choose K to *consist of discs* in the following sense:

$$K = \{(x_1, \sigma_1, x_2, \sigma_2) \in (\mathbb{R}^2_S)^2 : |x_1 - x_2|_h \leq r_{\sigma_1\sigma_2}\},$$

where $|.|_h$ is an arbitrary norm on $\mathbb{R}^2$ and $(r_{\sigma_1\sigma_2})_{\sigma_1,\sigma_2\in S}$ is a symmetric, measurable and bounded family of reals. In this case we define the enlargement $K_{+\epsilon}$ to be a set of the above form, where $r_{\sigma_1\sigma_2}$ is replaced by $r_{\sigma_1\sigma_2}+\epsilon$. We now simply set $K' := K_{+\epsilon}$ and $K'' := K'_{+\epsilon} = K_{+2\epsilon}$.

(c) If $(S, \mathcal{F}_S)$ is assumed to be a standard Borel space, then there is a metric on S, such that $\mathcal{F}_S$ is the corresponding Borel-σ-Algebra. Hence there is a measurable metric d on $(\mathbb{R}^2_S)^2$, and the d,ϵ-enlargement of an arbitrary set $A \subset (\mathbb{R}^2_S)^2$

$$A_{d,\epsilon} := \{(y_1,y_2) \in (\mathbb{R}^2_S)^2 : d((y_1,y_2),A) < \epsilon\}$$

is measurable. In this case we may set $K' = K_{d,\epsilon}$ and $K'' = (K')_{d,\epsilon} = K_{d,2\epsilon}$.

We note that in the above cases we often may replace K'' by K in the second inequality of (2.2). Here the following easy lemma is useful:

Lemma 3 *Let $K \subset (\mathbb{R}^2_S)^2$ be an $\vec{a}$-invariant standard set. For $\epsilon \to 0$*

(a) $\sup_{y_1\in\mathbb{R}^2_S} \int 1_{K_{\epsilon,\vec{a}}\setminus K}(y_1,y_2)dy_2 \to 0$ *in the situation of case (a) if we know that all $\vec{a}$-cross section of K are open intervals,*

(b) $\sup_{y_1\in\mathbb{R}^2_S} \int 1_{K_{+\epsilon}\setminus K}(y_1,y_2)dy_2 \to 0$ *in the situation of case (b).*

2.5 Smoothly approximable potentials

For convenience, in the following we will stick to the case of a hard core consisting of discs. The results show, how Definition 1 simplifies whenever U satisfies additional regularity properties such as smoothness or continuity.

Lemma 4 *Let $(U, z, \mathcal{Y}_0)$ be admissible with Ruelle bound, where $U : (\mathbb{R}^2_S)^2 \to \overline{\mathbb{R}}$ is an $\vec{a}$-invariant standard potential with a hard core K^U consisting of discs. U is $\vec{a}$-smoothly approximable if for every $\delta > 0$ there is a bpsi-function ψ_δ such that one of the following conditions is satisfied:*

(a) U is smooth and has ψ_δ-dominated $\vec{a}$-derivatives on $(K^U_{+\delta})^c$.

(b) U is bounded and $\vec{a}$-equicontinuous on $(K^U_{+\delta})^c$, and for some $R \in \mathbb{N}$ the set $\tilde{K} := \{(y_1,y_2) \in (\mathbb{R}^2_S)^2 : |y_1-y_2| \le R\}$ has one of the following properties:

(b1) U has ψ_δ-dominated $\vec{a}$-derivatives on $\tilde{K}^c$.

(b2) There is an $\vec{a}$-invariant standard potential $\tilde{U}$ such that $|U| \le \tilde{U}$ on $\tilde{K}^c$, $\tilde{U}$ is bounded and has ψ_δ-dominated $\vec{a}$-derivatives on $\tilde{K}^c$, and

$$\lim_{r\to\infty} \sup_{y_1\in\mathbb{R}^2_S} \int \tilde{U}(y_1,y_2)|y_1-y_2|^2 1_{\{|y_1-y_2|\ge r\}} dy_2 = 0.$$

For example, $(b1)$ holds trivially when U has finite range, and $(b2)$ includes the case that there are $\epsilon' > 0$ and $k \geq 0$ such that $|U(y_1, y_2)| \leq k/|y_1 - y_2|^{4+\epsilon'}$ for large $|y_1 - y_2|$. We note that the generalization of the preceding lemmas to more general hard cores is straightforward: Instead of imposing regularity conditions for U on $(K^U_{+\delta})^c$ for every $\delta > 0$ we just require it on some $\vec{a}$-invariant standard set $K \supset K^U$ such that there are measurable $\vec{a}$-enlargements K', K'' of K with $\int 1_{K'' \setminus K^U}(y_1, y_2) dy_2 < 1/(z\xi)$ for all $y_1 \in \mathbb{R}^2_S$, where ξ is a Ruelle bound. Finally we would like to show how to deal with discontinuous potentials by considering Potts type potentials as defined in Section 2.2.

Lemma 5 *Let S be a finite spin space and let $(U, z, \mathcal{Y}_0)$ be admissible with Ruelle bound, where U is a Potts type potential corresponding to a norm $|.|_h$ on $\mathbb{R}^2$ and a family of interactions $(\phi_{\sigma_1\sigma_2})_{\sigma_1,\sigma_2 \in S}$. If all the functions $\phi_{\sigma_1\sigma_2}$ are well behaved, then U is $\vec{a}$-smoothly approximable.*

Again we note that the ideas of the proof of Lemma 5 can be used to prove $\vec{a}$-smooth approximability for more general potentials, but for simplicity we restrict ourselves to the above case. Theorem 1 can now be seen to be an immediate consequence of Theorem 2 and Lemma 5. We only have to note that for nonnegative U $(U, z, \mathcal{Y})$ is admissible and admits a Ruelle bound (see Section 3.4, Lemma 7).

3 Setting

3.1 State space

We will use the notations $\mathbb{N} := \{0, 1, \ldots\}$, $\mathbb{R}_+ := [0, \infty[$, $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$, $r_1 \vee r_2 := \max\{r_1, r_2\}$, and $r_1 \wedge r_2 := \min\{r_1, r_2\}$ for $r_1, r_2 \in \mathbb{R}$. For sets A, B the cross sections of a subset $C \subset A \times B$ with respect to given elements $a \in A$ and $b \in B$ are denoted by

$$C(a) := \{b' \in B : (a, b') \in C\} \quad \text{and} \quad C(b) := \{a' \in A : (a', b) \in C\}.$$

The state space $\mathbb{R}^2_S := \mathbb{R}^2 \times S$ of a particle consists of the space of positions $\mathbb{R}^2$ and the spin space S. Usually we will denote particles by y, positions by x and spins by σ. Considering a model that does not include internal properties of particles we may simply set $S := \{0\}$. On $\mathbb{R}^2$ we consider the maximum norm $|.|$ and the Euclidean norm $|.|_2$. The Borel-σ-algebra $\mathcal{B}^2$ on $\mathbb{R}^2$ is induced by any of these norms. Let $\mathcal{B}^2_b$ be the set of all bounded Borel sets and λ^2 be the Lebesgue measure on $(\mathbb{R}^2, \mathcal{B}^2)$. Integration with respect to this measure will be abbreviated by $dx := d\lambda^2(x)$. Often we consider the centred squares

$$\Lambda_r := [-r, r[^2 \subset \mathbb{R}^2 \qquad (r \in \mathbb{R}_+).$$

For describing the spins of the particles let $(S, \mathcal{F}_S, \lambda_S)$ be a probability space such that the diagonal in $S \times S$ is measurable w.r.t $\mathcal{F}_S \otimes \mathcal{F}_S$. Integration with respect to λ_S will be abbreviated by $d\sigma := d\lambda_S(\sigma)$ and the same way we use $dy := d\lambda^2(x) d\lambda_S(\sigma)$ in the particle space. Sometimes we will apply functions of $\mathbb{R}^2$ to particles by simply ignoring their spins; for example $|y_1 - y_2|$ is defined to

be the distance between the positions of two particles $y_1, y_2 \in \mathbb{R}^2_S$. Similarly we may think of $\Lambda \subset \mathbb{R}^2$ as a set of particles by identifying this set with $\Lambda \times S \subset \mathbb{R}^2_S$.

We also want to consider bonds between particles. For a set X we define

$$E(X) := \{A \subset X : \#A = 2\}$$

to be the set of all bonds in X, where $\#$ denotes the cardinality of a set. A bond will be denoted by $xx' := \{x, x'\}$, where $x, x' \in X$ such that $x \neq x'$. Every symmetric function u on $X \times X$ can be considered a function on $E(X)$ via $u(xx') := u(x, x')$. For a bond set $B \subset E(X)$ (X, B) is a (simple) graph. The connectedness relation

$$x \overset{X,B}{\longleftrightarrow} x' \;:\Leftrightarrow\; \exists\, m \in \mathbb{N}, x_0, \ldots, x_m \in X : \; x = x_0, x' = x_m, \; x_{i-1}x_i \in B \;\; \forall\, i$$

defines an equivalence relation on X whose equivalence classes are called the B-clusters of X. Let

$$C_{X,B}(x) := \{x' \in X : x \overset{X,B}{\longleftrightarrow} x'\} \quad \text{and} \quad C_{X,B}(\Lambda) := \bigcup_{x' \in X \cap \Lambda} C_{X,B}(x')$$

denote the B-clusters of a point x and a set Λ respectively. Primarily we are interested in the case $X \subset \mathbb{R}^2_S$. On $E(\mathbb{R}^2_S)$ we consider the σ-algebra

$$\mathcal{F}_{E(\mathbb{R}^2_S)} := \{\{y_1y_2 \in E(\mathbb{R}^2_S) : (y_1, y_2) \in M\} : \; M \in (\mathcal{B}^2 \otimes \mathcal{F}_S)^2\}.$$

3.2 Configuration space

A set of particles $Y \subset \mathbb{R}^2_S$ is called

finite	if $\#Y < \infty$,	and
locally finite	if $\#(Y \cap \Lambda) < \infty$ for all $\Lambda \in \mathcal{B}^2_b$.	

The configuration space $\mathcal{Y}$ of particles is defined as the set of all locally finite subsets of $\mathbb{R}^2_S$. The elements of $\mathcal{Y}$ are called configurations of particles. For $Y \in \mathcal{Y}$ and $A \in \mathcal{B}^2 \otimes \mathcal{F}_S$ let

$Y_A := Y \cap A$	(restriction of Y to A),
$\mathcal{Y}_A := \{Y \in \mathcal{Y} : Y \subset A\}$	(set of all configurations in A), and
$N_A(Y) := \#Y_A$	(number of particles of Y in A).

The counting variables $(N_A)_{A \in \mathcal{B}^2 \otimes \mathcal{F}_S}$ generate a σ-algebra on $\mathcal{Y}$, which will be denoted by $\mathcal{F}_\mathcal{Y}$. For $\Lambda \in \mathcal{B}^2$ let $\mathcal{F}'_{\mathcal{Y},\Lambda}$ be the σ-algebra on $\mathcal{Y}_\Lambda$ obtained by restricting $\mathcal{F}_\mathcal{Y}$ to $\mathcal{Y}_\Lambda$, and let $\mathcal{F}_{\mathcal{Y},\Lambda} := e_\Lambda^{-1}\mathcal{F}'_{\mathcal{Y},\Lambda}$ be the σ-algebra on $\mathcal{Y}$ obtained from $\mathcal{F}'_{\mathcal{Y},\Lambda}$ by the restriction mapping $e_\Lambda : \mathcal{Y} \to \mathcal{Y}_\Lambda, Y \mapsto Y_\Lambda$. The tail σ-algebra or σ-algebra of the events far from the origin is defined by $\mathcal{F}_{\mathcal{Y},\infty} := \bigcap_{n \geq 1} \mathcal{F}_{\mathcal{Y},\Lambda_n^c}$. For configurations $Y, \bar{Y} \in \mathcal{Y}$ let $Y\bar{Y} := Y \cup \bar{Y}$. Let ν be the distribution of the Poisson point process on $(\mathcal{Y}, \mathcal{F}_\mathcal{Y})$, i.e.

$$\int \nu(dY) f(Y) = e^{-\lambda^2(\Lambda)} \sum_{k \geq 0} \frac{1}{k!} \int_{\Lambda^k} dy_1 \ldots dy_k \; f(\{y_i : 1 \leq i \leq k\})$$

for any $\mathcal{F}_{\mathcal{Y},\Lambda}$-measurable function $f : \mathcal{Y} \to \mathbb{R}_+$, where $\Lambda \in \mathcal{B}_b^2$. For $\Lambda \in \mathcal{B}_b^2$ and $\bar{Y} \in \mathcal{Y}$ let $\nu_\Lambda(.|\bar{Y})$ be the distribution of the Poisson point process in Λ with boundary condition $\bar{Y}$, i.e.

$$\int \nu_\Lambda(dY|\bar{Y}) f(Y) \; = \; \int \nu(dY) f(Y_\Lambda \bar{Y}_{\Lambda^c})$$

for any $\mathcal{F}_{\mathcal{Y}}$-measurable function $f : \mathcal{Y} \to \mathbb{R}_+$. It is easy to see that ν_Λ is a stochastic kernel from $(\mathcal{Y}, \mathcal{F}_{\mathcal{Y},\Lambda^c})$ to $(\mathcal{Y}, \mathcal{F}_{\mathcal{Y}})$.

The configuration space of bonds is the set of all locally finite bond sets:

$$\mathcal{E} := \{B \subset E(\mathbb{R}^2_S) : \#\{yy' \in B : yy' \subset \Lambda \times S\} < \infty \text{ for all } \Lambda \in \mathcal{B}_b^2\}.$$

On $\mathcal{E}$ the σ-algebra $\mathcal{F}_{\mathcal{E}}$ is defined to be generated by the counting variables $N_E : \mathcal{E} \to \mathbb{N}, B \mapsto \#(E \cap B)$ $\;(E \in \mathcal{F}_{E(\mathbb{R}^2_S)})$. For a countable set $E \in \mathcal{E}$ one can also consider the Bernoulli-σ-algebra $\mathcal{B}_E$ on $\mathcal{E}_E := \mathcal{P}(E) \subset \mathcal{E}$, which is defined to be generated by the family of sets $(\{B \subset E : e \in B\})_{e \in E}$. Given a family $(p_e)_{e\in E}$ of reals in $[0,1]$ the Bernoulli measure on $(\mathcal{E}_E, \mathcal{B}_E)$ is defined as the unique probability measure for that the events $(\{B \subset E : e \in B\})_{e\in E}$ are independent with probabilities $(p_e)_{e\in E}$. It is easy to check that the inclusion $(\mathcal{E}_E, \mathcal{B}_E) \to (\mathcal{E}, \mathcal{F}_{\mathcal{E}})$ is measurable. Thus any probability measure on $(\mathcal{E}_E, \mathcal{B}_E)$ can trivially be extended to $(\mathcal{E}, \mathcal{F}_{\mathcal{E}})$.

3.3 Gibbs measures

Let $U : (\mathbb{R}^2_S)^2 \to \overline{\mathbb{R}}$ be a potential and $z > 0$ an activity parameter. For finite configurations $Y, Y' \in \mathcal{Y}$ we consider the energy terms

$$H^U(Y) := \sum_{y_1 y_2 \in E(Y)} U(y_1, y_2) \quad \text{and} \quad W^U(Y, Y') := \sum_{y_1 \in Y} \sum_{y_2 \in Y'} U(y_1, y_2).$$

The last definition can be extended to locally finite configurations Y' whenever $W^U(Y, Y'_\Lambda)$ converges as $\Lambda \uparrow \mathbb{R}^2$ through the net $\mathcal{B}_b^2$. The Hamiltonian of a configuration $Y \in \mathcal{Y}$ in $\Lambda \in \mathcal{B}_b^2$ is given by

$$H^U_\Lambda(Y) := H^U(Y_\Lambda) + W^U(Y_\Lambda, Y_{\Lambda^c}) = \sum_{y_1 y_2 \in E_\Lambda(Y)} U(y_1, y_2),$$
$$\text{where} \quad E_\Lambda(Y) := \{y_1 y_2 \in E(Y) : y_1 y_2 \cap \Lambda \neq \emptyset\}.$$

The integral

$$Z^{U,z}_\Lambda(\bar{Y}) := \int \nu_\Lambda(dY|\bar{Y}) \, e^{-H^U_\Lambda(Y)} z^{\# Y_\Lambda}$$

is called the partition function in $\Lambda \in \mathcal{B}_b^2$ for the boundary condition $\bar{Y}_{\Lambda^c} \in \mathcal{Y}$. In order to ensure that the above objects are well defined and the partition function is finite and positive we need the following definition:

Definition 2 *A triple $(U, z, \mathcal{Y}_0)$ consisting of a potential $U : (\mathbb{R}^2_S)^2 \to \overline{\mathbb{R}}$, an activity parameter $z > 0$, and a set of boundary conditions $\mathcal{Y}_0 \in \mathcal{F}_{\mathcal{Y},\infty}$ is called admissible if for all $\bar{Y} \in \mathcal{Y}_0$ and $\Lambda \in \mathcal{B}_b^2$ the following holds: $W^U(\bar{Y}_\Lambda, \bar{Y}_{\Lambda^c})$ has a well defined value in $\overline{\mathbb{R}}$, and the partition function $Z^{U,z}_\Lambda(\bar{Y})$ is finite.*

If $(U,z,\mathcal{Y}_0)$ is admissible, $\Lambda\in\mathcal{B}^2_b$, and $\bar{Y}\in\mathcal{Y}_0$, then $W^U(Y_\Lambda,\bar{Y}_{\Lambda^c})\in\overline{\mathbb{R}}$ is well defined for every $Y\in\mathcal{Y}$, because $Y_\Lambda\bar{Y}_{\Lambda^c}\in\mathcal{Y}_0$. As a consequence the partition function $Z^{U,z}_\Lambda(\bar{Y})$ is well defined. Furthermore by definition it is finite, and by considering the empty configuration one can show that it is positive. The conditional Gibbs distribution $\gamma^{U,z}_\Lambda(.|\bar{Y})$ in $\Lambda\in\mathcal{B}^2_b$ with boundary condition $\bar{Y}\in\mathcal{Y}_0$ is thus well defined by

$$\gamma^{U,z}_\Lambda(A|\bar{Y}) := \frac{1}{Z^{U,z}_\Lambda(\bar{Y})}\int \nu_\Lambda(dY|\bar{Y})\, e^{-H^U_\Lambda(Y)} z^{\# Y_\Lambda} 1_A(Y) \quad \text{for} \quad A\in\mathcal{F}_\mathcal{Y}.$$

$\gamma^{U,z}_\Lambda$ is a probability kernel from $(\mathcal{Y}_0,\mathcal{F}_{\mathcal{Y}_0,\Lambda^c})$ to $(\mathcal{Y},\mathcal{F}_\mathcal{Y})$. Let

$$\begin{aligned}\mathcal{G}_{\mathcal{Y}_0}(U,z) := \{\mu\in&\mathcal{P}_1(\mathcal{Y},\mathcal{F}_\mathcal{Y}) :\ \mu(\mathcal{Y}_0)=1 \quad \text{and}\\ &\mu(A|\mathcal{F}_{\mathcal{Y},\Lambda^c}) = \gamma^{U,z}_\Lambda(A|.)\ \mu\text{-a.s. }\ \forall\, A\in\mathcal{F}_\mathcal{Y},\Lambda\in\mathcal{B}^2_b\}\end{aligned}$$

be the set of all Gibbs measures corresponding to U and z with whole weight on boundary conditions in $\mathcal{Y}_0$. It is easy to see that for any probability measure $\mu\in\mathcal{P}_1(\mathcal{Y},\mathcal{F}_\mathcal{Y})$ such that $\mu(\mathcal{Y}_0)=1$ we have the equivalence

$$\mu\in\mathcal{G}_{\mathcal{Y}_0}(U,z) \quad\Leftrightarrow\quad (\mu\otimes\gamma^{U,z}_\Lambda = \mu\,\forall\,\Lambda\in\mathcal{B}^2_b).$$

So for every $\mu\in\mathcal{G}_{\mathcal{Y}_0}(U,z)$, $f:\mathcal{Y}\to\mathbb{R}_+$ measurable and $\Lambda\in\mathcal{B}^2_b$ we have

$$\int\mu(dY)\,f(Y) = \int\mu(d\bar{Y})\int\gamma^{U,z}_\Lambda(dY|\bar{Y})\,f(Y). \tag{3.1}$$

We note that the hard core K^U of a potential U models the property that particles are not allowed to get too close to each other, i.e. for admissible $(U,z,\mathcal{Y}_0)$ and $\mu\in\mathcal{G}_{\mathcal{Y}_0}(U,z)$ we have

$$\mu(\{Y\in\mathcal{Y}:\exists y,y'\in Y: y\neq y',(y,y')\in K^U\}) = 0. \tag{3.2}$$

This is because for every $n\in\mathbb{N}$ and every boundary condition $\bar{Y}\in\mathcal{Y}_0$ we have

$$\gamma^{U,z}_{\Lambda_n}(\{Y\in\mathcal{Y}:\exists y,y'\in Y_{\Lambda_n}: y\neq y',(y,y')\in K^U\}|\bar{Y}) = 0,$$

as on the event considered in the last line the Hamiltonian $H^U_{\Lambda_n}(Y|\bar{Y})$ is infinite. Therefore (3.2) follows by using (3.1) and taking $n\to\infty$.

For admissible $(U,z,\mathcal{Y}_0)$ and a Gibbs measure $\mu\in\mathcal{G}_{\mathcal{Y}_0}(U,z)$ we define the correlation function $\rho^{U,\mu}$ by

$$\rho^{U,\mu}(Y) = e^{-H^U(Y)}\int\mu(d\bar{Y})\,e^{-W^U(Y,\bar{Y})}$$

for any finite configuration $Y\in\mathcal{Y}$. If there is a $\xi=\xi(U,z,\mathcal{Y}_0)\geq 0$ such that

$$\rho^{U,\mu}(Y) \leq \xi^{\# Y} \quad \text{for all finite } Y\in\mathcal{Y} \text{ and all } \mu\in\mathcal{G}_{\mathcal{Y}_0}(U,z), \tag{3.3}$$

then we call ξ a Ruelle bound for $(U,z,\mathcal{Y}_0)$. Actually we need this bound on the correlation function in the following way:

Lemma 6 *Let $(U, z, \mathcal{Y}_0)$ be admissible with Ruelle bound ξ. For every Gibbs measure $\mu \in \mathcal{G}_{\mathcal{Y}_0}(U, z)$ and every measurable function $f : (\mathbb{R}^2_S)^m \to \mathbb{R}_+$, $m \in \mathbb{N}$, we have*

$$\int \mu(dY) \sum_{y_1,\dots,y_m \in Y}^{\neq} f(y_1, \dots, y_m) \le (z\xi)^m \int dy_1 \dots dy_m \, f(y_1, \dots, y_m). \quad (3.4)$$

We use $\Sigma^{\neq}$ as a shorthand notation for a multiple sum such that the summation indices are assumed to be pairwise distinct.

3.4 Superstability and admissibility

Now we will discuss some conditions on potentials that imply that $(U, z, \mathcal{Y}_0)$ is admissible and has a Ruelle bound whenever the set of boundary conditions $\mathcal{Y}_0$ is suitably chosen. Apart from purely repulsive (i.e nonnegative) potentials such as the one considered in Theorem 1 we also want to consider superstable potentials in the sense of Ruelle, see [Ru1]. Therefore let $\Gamma_r := r + [-\frac{1}{2}, \frac{1}{2}[^2 \subset \mathbb{R}^2$ be the unit square centred at $r \in \mathbb{Z}^2$ and let $\mathbb{Z}^2(Y) := \{r \in \mathbb{Z}^2 : N_{\Gamma_r}(Y) > 0\}$ be the minimal set of lattice points such that the corresponding squares cover the configuration Y. A potential $U : (\mathbb{R}^2_S)^2 \to \overline{\mathbb{R}}$ is called superstable if there are real constants $a > 0$ and $b \ge 0$ such that

$$H^U(Y) \ge \sum_{r \in \mathbb{Z}^2(Y)} [a N_{\Gamma_r}(Y)^2 - b N_{\Gamma_r}(Y)]$$

for all finite configurations $Y \in \mathcal{Y}$. U is called lower regular if there is a decreasing function $\Psi : \mathbb{N} \to \mathbb{R}_+$ with $\sum_{r \in \mathbb{Z}^2} \Psi(|r|) < \infty$ such that

$$W^U(Y, Y') \ge - \sum_{r \in \mathbb{Z}^2(Y)} \sum_{s \in \mathbb{Z}^2(Y')} \Psi(|r - s|) \, [\frac{1}{2} N_{\Gamma_r}(Y)^2 + \frac{1}{2} N_{\Gamma_s}(Y')^2]$$

for all finite configurations $Y, Y' \in \mathcal{Y}$. So superstability and lower regularity give lower bounds on energies in terms of particle densities. In order to control these densities a configuration $Y \in \mathcal{Y}$ is said to be tempered if

$$\bar{s}(Y) := \sup_{n \in \mathbb{N}} s_n(Y) < \infty, \quad \text{where } s_n(Y) := \frac{1}{(2n+1)^2} \sum_{r \in \mathbb{Z}^2 \cap \Lambda_{n+1/2}} N^2_{\Gamma_r}(Y).$$

By $\mathcal{Y}_t$ we denote the set of all tempered configurations. We note that $\mathcal{Y}_t \in \mathcal{F}_{\mathcal{Y},\infty}$.

Lemma 7 *Let $z > 0$ and $U : (\mathbb{R}^2_S)^2 \to \overline{\mathbb{R}}$ be a potential function.*

(a) If $U \ge 0$, then $(U, z, \mathcal{Y})$ is admissible with Ruelle bound $\xi := 1$.

(b) If U is superstable and lower regular then $(U, z, \mathcal{Y}_t)$ is admissible and admits a Ruelle bound.

The first assertion is a straightforward consequence of the fact that all energy terms are nonnegative. For the second assertion see [Ru1].

3.5 Conservation of translational symmetry

We want to establish the conservation of $\vec{\tau}$-translational symmetry for a given admissible triple $(U, z, \mathcal{Y}_0)$ and a translation $\vec{\tau} \in \mathbb{R}^2$. It suffices to show that for every $\delta > 0$ and every cylinder event $D \in \mathcal{F}_{\mathcal{Y},\Lambda_m}$ $(m \in \mathbb{N})$ there is a natural $n \geq m$ such that we have

$$\gamma_{\Lambda_n}^{U,z}(D + \vec{\tau}|\bar{Y}) \,+\, \gamma_{\Lambda_n}^{U,z}(D - \vec{\tau}|\bar{Y}) \,\geq\, 2\gamma_{\Lambda_n}^{U,z}(D|\bar{Y}) \,-\, \delta \quad \text{for all } \bar{Y} \in \mathcal{Y}_0. \tag{3.5}$$

Indeed, let $\mu \in \mathcal{G}_{\mathcal{Y}_0}(U, z)$, then integrating (3.5) with respect to μ and applying (3.1) gives $\mu(D+\vec{\tau})+\mu(D-\vec{\tau}) \geq 2\mu(D)-\delta$ for all $\delta > 0$ and $D \in \mathcal{F}_{\mathcal{Y},\Lambda_m}$ $(m \in \mathbb{N})$. Letting $\delta \to 0$, Lemma 1 shows the invariance of μ under the translation by $\vec{\tau}$, because the cylinder events form an algebra which generates the σ-algebra $\mathcal{F}_{\mathcal{Y}}$. For the proof of Theorem 2 our goal thus will be to establish an inequality similar to (3.5). We further note that the group $\mathbb{R}\vec{a}$ is generated by the set $\{\tau\vec{a} : \tau \in [0, 1/2]\}$. Thus we only have to consider translations of this special form in order to establish the $\vec{a}$-invariance of a set of Gibbs measures.

3.6 Concerning measurability

We will consider various types of random objects, all of which have to be shown to be measurable with respect to the considered σ-algebras. However, we will not prove measurability of every such object in detail. Instead, we will now give a list of operations that preserve measurability.

Lemma 8 *Let $Y, Y' \in \mathcal{Y}$, $B, B' \in \mathcal{E}$, $y \in \mathbb{R}^2_S$, $x \in \mathbb{R}^2$, and $p \in \Omega$ be variables, where $(\Omega, \mathcal{F})$ is a measurable space. Let $f : \Omega \times (\mathbb{R}^2_S) \to \mathbb{R}$ and $g : \Omega \times E(\mathbb{R}^2_S) \to \mathbb{R}$ be measurable. Then the following functions of the given variables are measurable with respect to the considered σ-algebras:*

$$\sum_{y' \in Y} f(p, y'), \quad Y \cap Y', \quad Y \cup Y', \quad Y \setminus Y', \quad Y + x, \tag{3.6}$$

$$\sum_{b' \in B} g(p, b'), \quad B \cap B', \quad B \cup B', \quad B \setminus B', \quad B + x, \tag{3.7}$$

$$\inf_{y' \in Y} f(p, y'), \quad C_{Y,B}(y), \quad E(Y). \tag{3.8}$$

Using this lemma and well known theorems, such as the measurability part of Fubini's theorem, we can check the measurability of all objects considered.

4 Proof of the lemmas from Sections 2 and 3

4.1 Conservation of symmetries: Lemma 1

We first note that $\mu\circ\tau^{-1} = \mu$ easily implies (2.1), where we indeed have equality. For the other implication let us assume (2.1). By the monotone class theorem this inequality immediately extends to all $A \in \mathcal{F}$. Thus for all $D \in \mathcal{F}$ and $k \in \mathbb{Z}$

$$\mu(\tau^{k+1}D) \,+\, \mu(\tau^{k-1}D) \,\geq\, 2\mu(\tau^k D),$$

i.e. the sequence $(\mu(\tau^k D))_{k\in\mathbb{Z}}$ is convex. But μ is a probability measure, so the sequence is bounded, and thus it has to be constant. In particular we get $\mu(\tau^{-1}D) = \mu(D)$. As $D \in \mathcal{F}$ was arbitrary the result follows.

4.2 Measurable enlargements: Lemmas 2 and 3

Let A be as described in Lemma 2 and let $\epsilon > 0$. By Fubini's theorem the function $f : (\mathbb{R}^2_S)^2 \to \mathbb{R}_+,\ f(y_1, y_2) := \lambda(A(y_1, y_2, \vec{a}) \cap\,]-\epsilon, \epsilon[\,)$ is measurable and by (2.3) for all $y_1, y_2 \in \mathbb{R}^2_S$ we have

$$f(y_1, y_2) > 0 \quad \Leftrightarrow \quad A(y_1, y_2, \vec{a}) \cap\,]-\epsilon, \epsilon[\, \neq \emptyset \quad \Leftrightarrow \quad (y_1, y_2) \in A_{\epsilon,\vec{a}}.$$

This shows $A_{\epsilon,\vec{a}} = \{f > 0\}$ to be measurable. The second statement of Lemma 2 is an immediate consequence of the fact that a Borel set containing a nonempty open set has positive Lebesgue-measure.

For the proof of Lemma 3 (a) let all $\vec{a}$-cross sections of K be open intervals. Then the $\vec{a}$-cross sections $(K_{\epsilon,\vec{a}} \setminus K)(y_1, y_2, \vec{a})$ are either empty or the union of two intervals of length ϵ. Furthermore, as K is of bounded range there is a real $r > 0$ such that for every $(y_1, y_2) \in K_{\epsilon,\vec{a}}$ we have $|y_1 - y_2| \leq r$. The supremum in (a) can thus be estimated by

$$\sup_{r_1, r_1' \in \mathbb{R}} \sup_{\sigma_1,\sigma_2\in S} \int dr_2' 1_{\{|r_1'-r_2'|\leq r\}} \int dr_2 1_{\{r_2 \in (K_{\epsilon,\vec{a}}\setminus K)(r_1, r_1', \sigma_1, r_2', \sigma_2)\}} \,\leq\, 2r \cdot 2\epsilon.$$

For part (b) let K be a standard set consisting of discs and let $r > 0$ be a bound for the corresponding family $(r_{\sigma_1\sigma_2})_{\sigma_1,\sigma_2\in S}$. The supremum in (b) can be estimated by

$$\sup_{x_1\in\mathbb{R}^2} \sup_{\sigma_1,\sigma_2\in S} \int 1_{\{r_{\sigma_1\sigma_2} < |x_1 - x_2|_h \leq r_{\sigma_1\sigma_2}+\epsilon\}} dx_2 \,\leq\, \lambda^2(\{r < |.|_h < r + \epsilon\}).$$

4.3 Smooth or continuous potentials: Lemma 4

We set $K := K^U_{+\delta}$, $K' := K_{+\epsilon}$, $K'' = K'_{+\epsilon}$, where $\epsilon, \delta > 0$ are so small that

$$c \,:=\, 1/(z\xi) - \sup_{y_1\in\mathbb{R}^2_S} \int 1_{K''\setminus K^U}(y_1, y_2)\, dy_2 \,>\, 0,$$

where ξ is a Ruelle bound for $(U, z, \mathcal{Y}_0)$. (This is possible by Lemma 3.) In case (a) we are done setting $\psi := \psi_\delta$, $\bar{U} := U$ and $u := 0$. In case (b1) let $U_1 := U$

and in case (b2) let $U_1 := \tilde{U}$. Without loss of generality we may assume that $R \geq 1$ and $K \subset \tilde{K}$, and furthermore

$$\sup_{y_1 \in \mathbb{R}_S^2} \int 2\tilde{U}(y_1, y_2)|y_1 - y_2|^2 1_{\tilde{K}^c}(y_1, y_2)\, dy_2 \;<\; \frac{c}{2}$$

in case (b2). In both cases U_1 serves as an $\vec{a}$-smooth approximation of U on $\tilde{K}^c$. We note that U_1 is bounded and has ψ_δ-dominated $\vec{a}$-derivatives on $\tilde{K}^c$, which also implies that $\partial_{\vec{a}}^2 U_1$ and $\partial_{\vec{a}} U_1$ are bounded on $\tilde{K}^c$. Let

$$C := \{(y_1, y_2) \in (\mathbb{R}_S^2)^2 : |y_1 - y_2| \leq R + 1\} \setminus K.$$

For $\delta' > 0$ let $f_{\delta'} : \mathbb{R} \to \mathbb{R}_+$ be a symmetric smooth probability density with support in $]-\delta', \delta'[$, e.g. $f_{\delta'}(t) := \frac{1}{c_{\delta'}} 1_{]-\delta',\delta'[}(t) e^{-(1-t^2/\delta'^2)^{-1}}$, where $c_{\delta'}$ is a normalizing constant. Then

$$U_2(x_1, \sigma_1, x_2, \sigma_2) \;:=\; \int dt\, f_{\delta'}(t) U(x_1, \sigma_1, x_2 - t\vec{a}, \sigma_2)$$

is an $\vec{a}$-smooth approximation of U on C. If δ' is small enough, then

$$|U_2(y_1, y_2) - U(y_1, y_2)| \;<\; \frac{c}{16(R+1)^2} \qquad \text{for } (y_1, y_2) \in C$$

by the $\vec{a}$-equicontinuity of U. Let $g : (\mathbb{R}_S^2)^2 \to [0, 1]$ be an $\vec{a}$-smooth function with $g(y_1, y_2) = 0$ for $|y_1 - y_2| \leq R$, $g(y_1, y_2) = 1$ for $|y_1 - y_2| \geq R + 1$ and such that the $\vec{a}$-derivatives $\partial_{\vec{a}} g$ and $\partial_{\vec{a}}^2 g$ are bounded. Now we can define $\bar{U}, u : K^c \to \mathbb{R}$ by $\bar{U} := (1-g)(U_2 + c') + gU_1$ and $u := \bar{U} - U$. It is easy to verify that the constructed objects have all the properties described in Definition 1 in both cases (b1) and (b2).

4.4 Potts type potentials: Lemma 5

We first consider a well behaved function $\phi : \mathbb{R}_+ \to \overline{\mathbb{R}}$ (with respect to given reals $0 \leq r_0 < \ldots < r_n$, $n \geq 0$) and show how to decompose ϕ into a continuous part $\bar{\phi}$ and a small part φ. For $s, \epsilon > 0, m \in \mathbb{R}$ we define $h_{s,m,\epsilon} : \mathbb{R}_+ \to \mathbb{R}$ such that the graph of $h_{s,m,\epsilon}$ looks like $\bigwedge$, where (s, m) is the topmost point and the angle is determined by ϵ, i.e. $h_{s,m,\epsilon}(r) := m - (m/\epsilon)|r - s|$. Defining

$$\bar{\phi} := \phi \vee \bigvee_{i=1}^{n} h_{r_i, m_i, \epsilon}, \qquad \text{where } \; m_i := \Big(\phi(r_i) \vee \lim_{r \to r_i+} \phi(r) \vee \lim_{r \to r_i-} \phi(r)\Big) + 1,$$

for a given $\epsilon > 0$, we see that $\bar{\phi}$ is continuous on $]r_0, \infty[$. Furthermore $\varphi := \bar{\phi} - \phi$ satisfies $\varphi \geq 0$, $\int (\varphi(|y|_h) \wedge 1)|y|^2 1_{\{|y|_h > r_0\}} dy < \infty$, and $\int (\varphi(|y|_h) \wedge 1) 1_{\{|y|_h > r_0\}} dy$ is arbitrarily small if only $\epsilon > 0$ is chosen small enough.

Now let U be a Potts type potential corresponding to a norm $|.|_h$ and a family of well behaved interactions $(\phi_{\sigma_1\sigma_2})_{\sigma_1,\sigma_2 \in S}$, where S is a finite spin space. As above we decompose every $\phi_{\sigma_1\sigma_2}$ into a continuous part $\bar{\phi}_{\sigma_1\sigma_2}$ and a small part $\varphi_{\sigma_1\sigma_2}$, where the $\epsilon > 0$ entering the above construction is chosen sufficiently

small. Now let $U_c(y_1, y_2) := \bar{\phi}_{\sigma_1\sigma_2}(|x_1 - x_2|_h)$ and $u_c := U_c - U$. We observe that U_c is of the form described in Lemma 4 (b1): We simply choose $\psi = 0$ and $\tilde{K}$ so big such that $U_c = 0$ on $\tilde{K}^c$. We note that U_c is bounded and $\vec{a}$-equicontinuous in $K^U_{+\delta}$ for every $\delta > 0$. As in the proof of Lemma 4 we thus find a decomposition of U_c into suitable potentials $\bar{U}$ and u. Then $\bar{U}$ and $u + u_c$ give a decomposition of U into a smooth part and a small part as required.

4.5 Property of the Ruelle bound: Lemma 6

For every $n \in \mathbb{N}$, every measurable $g : \mathcal{Y}_{\Lambda_n} \to \mathbb{R}_+$ and every $\bar{Y} \in \mathcal{Y}_0$ we have

$$\int \nu_{\Lambda_n}(dY|\bar{Y}) \sum^{\neq}_{y_1,\ldots,y_m \in Y_{\Lambda_n}} f(y_1, \ldots, y_m)\, g(Y)$$
$$= \int_{{\Lambda_n}^m} dy_1 \ldots dy_m\, f(y_1, \ldots, y_m) \int \nu_{\Lambda_n}(dY'|\bar{Y})\, g(\{y_1, \ldots, y_m\}Y').$$

Combining this with (3.1), the definition of the conditional Gibbs distribution and the definition of the correlation function we get

$$\int \mu(dY) \sum^{\neq}_{y_1,\ldots,y_m \in Y_{\Lambda_n}} f(y_1, \ldots, y_m)$$
$$= \int \mu(d\bar{Y}) \frac{1}{Z^{U,z}_{\Lambda_n}(\bar{Y})} \int \nu_{\Lambda_n}(dY|\bar{Y}) \sum^{\neq}_{y_1,\ldots,y_m \in Y_{\Lambda_n}} f(y_1, \ldots, y_m)\, e^{-H^U_{\Lambda_n}(Y)} z^{\# Y_{\Lambda_n}}$$
$$= \int_{{\Lambda_n}^m} dy_1 \ldots dy_m\, f(y_1, \ldots, y_m)\, z^m\, \rho^{U,\mu}(\{y_1, \ldots, y_m\}).$$

Now we use (3.3) to estimate the correlation function by the Ruelle bound ξ. Letting $n \to \infty$ the assertion follows from the monotone limit theorem.

4.6 Measurability: Lemma 8

Details concerning measurability of functions of point processes can be found in [DV] or [MKM], for example. The first part of (3.6) is the measurability part of Campbell's theorem. For the rest of (3.6) it suffices to observe that we have $N_A(Y \cap Y') = \sum\limits_{y\in Y} \sum\limits_{y'\in Y'} 1_{\{y=y'\in A\}}$, $N_A(Y \setminus Y') = N_A(Y) - N_A(Y \cap Y')$, $N_A(Y \cup Y') = N_A(Y) + N_A(Y' \setminus Y)$ and $N_A(Y + x) = \sum\limits_{y\in Y} 1_A(y + x)$ for all $A \in \mathcal{B}^2_b \otimes \mathcal{F}_S$. (3.7) can be proved similarly. For (3.8) we note that $\inf\limits_{y'\in Y} f_1(p, y') < c$ $\Leftrightarrow \sum\limits_{y'\in Y} 1_{\{f_1(p,y')<c\}} \geq 1$ for all $c \in \mathbb{R}$, $N_A(C_{Y,B}(y)) = \sum\limits_{y'\in Y} 1_{\{y'\in C_{Y,B}(x), y'\in A\}}$ for all $A \in \mathcal{B}^2_b \otimes \mathcal{F}_S$, $y' \in C_{Y,B}(y) \Leftrightarrow \sum\limits_{m\geq 0} \sum\limits_{y_0,\ldots,y_m\in Y} 1_{\{y=y_0, y'=y_m\}} \prod\limits_{i=1}^{m} 1_{\{y_i y_{i+1}\in B\}} \geq 1$ for all $y' \in \mathbb{R}^2_S$ and $N_L(E(Y)) = \frac{1}{2}\sum_{y_1,y_2\in Y} 1_{\{y_1 y_2 \in L\}}$ for all $L \in \mathcal{F}_{E(\mathbb{R}^2_S)}$. Using these relations, the measurability of the terms in (3.8) follows easily. Note that we made repeated use of the fact that the diagonal is measurable in $S \times S$.

5 Proof of Theorem 2: Main steps

5.1 Basic constants

Let $(U, z, \mathcal{Y}_0)$ be admissible with Ruelle bound ξ, where $U : (\mathbb{R}^2_S)^2 \to \overline{\mathbb{R}}$ is an $\vec{a}$-invariant, $\vec{a}$-smoothly approximable standard potential. There is an $\vec{a}$-invariant standard set $K \supset K^U$ with $\vec{a}$-enlargements K', K'', a bpsi-function ψ, and measurable symmetric $\vec{a}$-invariant functions $\bar{U} : (\mathbb{R}^2_S)^2 \to \overline{\mathbb{R}}$ and $u : (\mathbb{R}^2_S)^2 \to \mathbb{R}$ such that $U = \bar{U} - u$ and $u \geq 0$ on $(\mathbb{R}^2_S)^2$, $u = 0$ on K, $\bar{U}$ has ψ-dominated $\vec{a}$-derivatives on K^c, and $\tilde{u} = 1 - e^{-u}$ satisfies

$$\begin{aligned} c_u &:= \sup_{y_1 \in \mathbb{R}^2_S} \int \tilde{u}(y_1, y_2)|y_1 - y_2|^2 dy_2 \,<\, \infty \qquad \text{and} \\ c_\xi &:= \sup_{y_1 \in \mathbb{R}^2_S} \int (1_{K'' \setminus K^U} + \tilde{u})(y_1, y_2) dy_2 \,<\, \frac{1}{z\xi}. \end{aligned} \tag{5.1}$$

Note that above we defined $\bar{U}$ and u also on K. By symmetry we may suppose that the direction of the translations is $\vec{a} = \vec{e} := (1, 0)$, and w.l.o.g. we may assume that

$$K \supset \{(x_1, \sigma_1, x_2, \sigma_2) \in (\mathbb{R}^2_S)^2 : x_1 = x_2\}.$$

Let $f_K : (\mathbb{R}^2_S)^2 \to [0, 1]$ be a measurable function such that

$$f_K = 0 \text{ on } K,\ f_K = 1 \text{ on } (K'')^c,\ f_K \text{ is } \vec{e}\text{-smooth, and } \partial_{\vec{e}} f_K \text{ is bounded,}$$

where $\partial_{\vec{e}} f_K$ is the $\vec{e}$-derivative with respect to the second spatial component. For the construction of such a function we introduce $\tilde{f} : (\mathbb{R}^2_S)^2 \to \mathbb{R}$, $\tilde{f} := 1_{(K')^c}$, and choose an infinitely often differentiable function $f_\epsilon : \mathbb{R} \to \mathbb{R}_+$ which is a probability density with support in $] - \epsilon, \epsilon[$. Then the function $f_K(y', y) := \int dt \tilde{f}(y', y - t\vec{e}) f_\epsilon(t)$ has the desired properties if $\epsilon > 0$ is chosen small enough. Furthermore we need the following constants:

$$\begin{aligned} c_\psi &:= \|\psi\| \vee \sup_{y_1 \in \mathbb{R}^2_S} \int dy_2\, \psi(y_1, y_2)(|y_1 - y_2|^2 \vee 1), \\ c_K &:= \sup\{|y_1 - y_2| : (y_1, y_2) \in K''\}, \qquad \text{and} \qquad c_f \,:=\, \|\partial_{\vec{e}} f_K\|. \end{aligned} \tag{5.2}$$

These constants are finite as ψ is a bpsi-function, K'' has bounded range, and $\partial_{\vec{e}} f_K$ is bounded. On $\mathbb{R}^2_S$ we consider the partial order $\leq_{\vec{e}}$ defined by

$$(r_1, r_2, \sigma) \,\leq_{\vec{e}}\, (r_1', r_2', \sigma') \quad :\Leftrightarrow \quad r_1 \leq r_1', r_2 = r_2', \sigma = \sigma'.$$

In order to show the conservation of $\vec{e}$-translational symmetry we fix a Gibbs measure $\mu \in \mathcal{G}_{\mathcal{Y}_0}(U, z)$, a cylinder event $D \in \mathcal{F}_{\mathcal{Y}, \Lambda_{n'-1}}$, where $n' \in \mathbb{N}$, a real $\delta \in]0, 1/2[$, and a translation distance parameter $\tau \in [0, 1/2]$, see subsection 3.5. We will ignore dependence on any of the above parameteres in our notations.

5.2 Decomposition of μ and the bond process

We consider the bond set $E_n(Y) := E_{\Lambda_n}(Y) = \{y_1y_2 \in E(Y) : y_1y_2 \cap \Lambda_n \neq \emptyset\}$ for $n \in \mathbb{N}$ and $Y \in \mathcal{Y}$. On $(\mathcal{E}_{E_n(Y)}, \mathcal{B}_{E_n(Y)})$ we introduce the Bernoulli measure $\pi_n(.|Y)$ with bond probabilities

$$(\tilde{u}(b))_{b\in E_n(Y)} \quad \text{where} \quad \tilde{u}(b) := 1 - e^{-u(b)},$$

using the shorthand notation $u(y_1y_2) := u(y_1, y_2)$ for $y_1, y_2 \in \mathbb{R}^2_S$. We note that $0 \le \tilde{u}(b) < 1$ for all $b \in E_n(Y)$ as $0 \le u < \infty$. As remarked earlier $\pi_n(.|Y)$ can be extended to a probability measure on $(\mathcal{E}, \mathcal{F}_\mathcal{E})$. For all $D \in \mathcal{F}_\mathcal{E}$ $\pi_n(D|.)$ is $\mathcal{F}_\mathcal{Y}$-measurable, so π_n is a probability kernel from $(\mathcal{Y}, \mathcal{F}_\mathcal{Y})$ to $(\mathcal{E}, \mathcal{F}_\mathcal{E})$.

Lemma 9 *Let $n \in \mathbb{N}$. We have*

$$\mu \otimes \nu_{\Lambda_n}(G_n) = 1 \ \ \textit{and} \ \ \mu(G_n) = 1 \ \ \textit{for} \ \ G_n := \Big\{Y \in \mathcal{Y} : \sum_{b\in E_n(Y)} \tilde{u}(b) < \infty\Big\}.$$

For $Y \in G_n$ every bond set is finite $\pi_n(.|Y)$-a.s. by Borel-Cantelli, so $\pi_n(.|Y) \ll \pi'_n(.|Y)$, where $\pi'_n(.|Y)$ denotes the counting measure on $(\mathcal{E}_{E_n(Y)}, \mathcal{B}_{E_n(Y)})$ concentrated on finite bond sets. Again, π'_n can be considered as a probability kernel from $(\mathcal{Y}, \mathcal{F}_\mathcal{Y})$ to $(\mathcal{E}, \mathcal{F}_\mathcal{E})$. We have

$$\frac{d\pi_n(.|Y)}{d\pi'_n(.|Y)}(B) = \prod_{b\in B} \tilde{u}(b) \prod_{b\in E_n(Y)\setminus B} (1 - \tilde{u}(b)) = e^{-H^u_{\Lambda_n}(Y)} \prod_{b\in B} (e^{u(b)} - 1),$$

so for every $Y \in G_n$ the Hamiltonian $H^u_{\Lambda_n}(Y)$ is finite, and thus the decomposition of the potential gives a corresponding decomposition of the Hamiltonian

$$H^U_{\Lambda_n}(Y) = H^{\bar{U}}_{\Lambda_n}(Y) - H^u_{\Lambda_n}(Y).$$

Using (3.1) we conclude that for every $\mathcal{F}_\mathcal{Y} \otimes \mathcal{F}_\mathcal{E}$-measurable function $f \ge 0$

$$\begin{aligned} \int d\mu \otimes \pi_n\, f \;=\; \int \mu(d\bar{Y}) \frac{1}{Z^{U,z}_{\Lambda_n}(\bar{Y})} \int \nu_{\Lambda_n} \otimes \pi'_n(dY, dB|\bar{Y}) \\ z^{\#Y_{\Lambda_n}} e^{-H^{\bar{U}}_{\Lambda_n}(Y)} \prod_{b\in B} (e^{u(b)} - 1) f(Y, B). \end{aligned} \tag{5.3}$$

Here by Lemma 9 on both sides we have $Y \in G_n$ with probability one, thus the equality follows from the above decomposition. If f does not depend on B at all, the integral on the left hand side of (5.3) is just the μ-expectation of f, as π_n is a probability kernel, and from the right hand side we learn that the perturbation u of the $\vec{e}$-smooth potential $\bar{U}$ can be encoded in a bond process B such that the perturbation affects only those pairs of particles with $y_1y_2 \in B$.

5.3 Generalized translation and good configurations

For integers n, R such that $n > R \ge n'$ we define the functions $q : \mathbb{R}_+ \to \mathbb{R}$, $Q : \mathbb{R}_+ \to \mathbb{R}$, $r : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ and $\tau_{R,n} : \mathbb{R} \to \mathbb{R}$ by

$$q(s) := \frac{1}{1 \vee (s \log(s))}, \qquad Q(k) := \int_0^k q(s)ds,$$

$$r(s,k) := \int_{(s\vee 0)\wedge k}^k \frac{q(s')}{Q(k)} ds', \qquad \tau_{R,n}(s) := \tau\, r(s - R, n - R).$$

For a sketch of the graph of $\tau_{R,n}$ see Figure 2. Some properties of $\tau_{R,n}$ are:

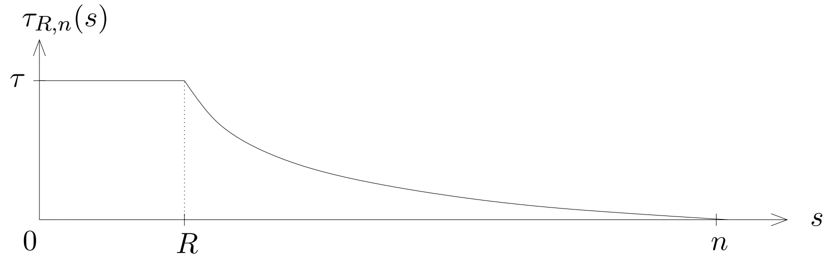

Figure 2: Graph of $\tau_{R,n}$

$$\tau_{R,n}(s) = \tau \text{ for } s \leq R, \quad \tau_{R,n}(s) = 0 \text{ for } s \geq n, \quad \tau_{R,n} \text{ is decreasing.} \tag{5.4}$$

Now $T_{R,n}(y) := y + \tau_{R,n}(|y|)\vec{e}$ defines a transformation on $\mathbb{R}^2_S$. This transformation can also be viewed as a transformation on $\mathcal{Y}$, such that every point y of a configuration Y is translated the distance $\tau_{R,n}(|y|)$ in direction $\vec{e}$. We would like to use $T_{R,n}$ as a tool for our proof just as in [FP1] and [FP2], but in order to deal with the hard core and the perturbation u, which is encoded in the bond process, we have to allow the transformation of a particle y to depend on the configuration of particles in the neighborhood of y and the configuration of bonds joining y to other particles. We thus aim to construct a transformation

$$\mathfrak{T}_{R,n} : \mathcal{Y} \times \mathcal{E} \to \mathcal{Y} \times \mathcal{E}$$

that is required to have the following properties:

(1) For $B \subset E(Y)$ the transformed configuration $(\tilde{Y}, \tilde{B}) = \mathfrak{T}_{R,n}(Y, B)$ is constructed by translating every particle $y \in Y$ by a certain distance in direction $\vec{e}$, and by translating bonds along with the corresponding particles.

(2) Particles in the inner region $\Lambda_{n'-1}$ are translated by $\tau\vec{e}$, and particles in the outer region ${\Lambda_n}^c$ are not translated at all.

(3) Particles connected by a bond in B are translated the same distance.

(4) $\mathfrak{T}_{R,n}$ is bijective, and the density of the transformed process with respect to the untransformed process under the measure $\nu \otimes \pi'_n$ can be calculated.

(5) We have suitable estimates on this density and on $H^{\bar{U}}_{\Lambda_n}(\tilde{Y}) - H^{\bar{U}}_{\Lambda_n}(Y)$. For the last assumption we need particles within hard core distance to remain within hard core distance and particles at larger distance to remain at larger distance.

Property (2) implies that the translation of the chosen cylinder event D is the same as the transformation of D by $\mathfrak{T}_{R,n}$. Properties (3)-(5) are chosen with a view to the right hand side of (5.3): If $\mathfrak{T}_{R,n}$ has these properties then the

density of the transformed process with respect to the untransformed process under the measure $\mu \otimes \pi_n$ can be estimated. Therefore a transformation with these properties seems to be the right tool for proving (3.5). However, in general it is impossible to construct a transformation with all the given properties. For example properties (2) and (5) cannot both be satisfied if Y is a configuration of densely packed hard-core particles, or properties (2) and (3) cannot both be satisfied if the inner and the outer region are connected by bonds. Similar problems arise for some of the other properties, so we will content ourselves with a transformation satisfying the above properties only for configurations (Y, B) from a set of good configurations $G_{R,n}$, which will be shown to have probability close to 1 for suitably chosen R and n in Lemma 15. We define $G_{R,n}$ to be of the form

$$G_{R,n} := \big\{(Y,B) \in \mathcal{Y} \times \mathcal{E} : \; B \subset E(Y), r_{n'}^{Y,B_+} < R, \sum_{i=1}^{5} \Sigma_i < \delta\big\}, \tag{5.5}$$

where $\delta \in]0, 1/2[$ is the constant chosen in section 5.1. The functions $\Sigma_i = \Sigma_i(R, n, Y, B)$ will be defined whenever we want good configurations to have certain properties, see (6.8), (6.22) and (6.23). The condition involving $r_{n'}^{Y,B_+}$ is meant to ensure that both the particle density and the number of bonds is not too high. More precisely for $Y \in \mathcal{Y}$ and $B \subset E(Y)$ let

$$B_+ := B \cup \{y_1 y_2 \in E(Y) : (y_1, y_2) \in K''\}$$

be the enlargement of B by additional bonds between particles that are close to each other. We then define

$$r_{n'}^{Y,B_+} := \sup\{|y'| : y' \in C_{Y,B_+}(\Lambda_{n'})\}$$

to be the range of the B_+-cluster of the inner region $\Lambda_{n'}$. For $y \in Y$ let

$$\tau_{R,n}^{\wedge,Y,B_+}(y) := \min\{\tau_{R,n}(|y'|) : y' \in C_{Y,B_+}(y)\}.$$

As Y_{Λ_n} is finite and $\tau_{R,n}(|.|) = 0$ on Λ_n^c by (5.4), this minimum is attained. By definition

$$(Y,B) \in G_{R,n}, \; y \in Y_{\Lambda_{n'}} \quad \Rightarrow \quad \tau_{R,n}^{\wedge,Y,B_+}(y) = \tau. \tag{5.6}$$

5.4 Modifying the generalized translation

We now define a transformation $\mathfrak{T}_{R,n}$ with the properties described in the last section. As $n > R \geq n'$ are fixed throughout this section, we usually will omit the dependence on n and R in our notations. With a view to properties (1) and (3) we define the transformation

$$\mathfrak{T}_{R,n} : \mathcal{Y} \times \mathcal{E} \to \mathcal{Y} \times \mathcal{E}, \quad \mathfrak{T}_{R,n}(Y,B) := (\mathfrak{T}_{R,n}^B(Y), \mathfrak{T}_{R,n}^Y(B))$$

$$\text{by} \quad \mathfrak{T}_{R,n}^B(Y) := \bigcup_{k=0}^{m(Y,B)} (C_k^{Y,B} + \tau_k^{Y,B} \vec{e}) = \{y + t^{Y,B}(y)\vec{e} : y \in Y\}$$

$$\text{and} \quad \mathfrak{T}_{R,n}^Y(B) := \{(y + t^{Y,B}(y)\vec{e})(y' + t^{Y,B}(y')\vec{e}) : yy' \in B\}$$

if B is a subset of $E(Y)$, and $\mathfrak{T}^B_{R,n} = id$ and $\mathfrak{T}^Y_{R,n} = id$ otherwise. Here $\big(C^{Y,B}_k\big)_{0\le k\le m(Y,B)}$ is a certain partition of Y, where every $C^{Y,B}_k$ is a union of B-clusters. $\tau^{Y,B}_k$ is the translation distance of all points in $C^{Y,B}_k$, and the

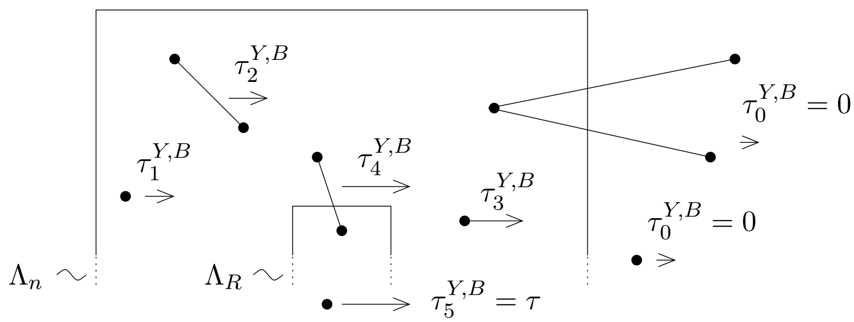

Figure 3: Every set $C^{Y,B}_k$ is translated by $\tau^{Y,B}_k\vec{e}$

translation distance function $t^{Y,B} : Y \to \mathbb{R}$ is defined by $t^{Y,B}(y) := \tau^{Y,B}_k$ for $y \in C^{Y,B}_k$. We are left to identify the points of $C^{Y,B}_k$ and their translation distances $\tau^{Y,B}_k$. In our construction we would like to ensure that the sets $C^{Y,B}_k$ are ordered in a way such that

$$\tau^{Y,B}_0 \le \tau^{Y,B}_1 \le \ldots \le \tau^{Y,B}_m. \tag{5.7}$$

This relation will be an important tool for showing the bijectivity of the transformation as required in property (4) of the last subsection. As required in (5) we also would like to have

$$y_1, y_2 \in Y,\ (y_1,y_2) \in K \ \Rightarrow\ t^{Y,B}(y_1) \ =\ t^{Y,B}(y_2), \tag{5.8}$$

$$y_1, y_2 \in Y,\ (y_1,y_2) \notin K \ \Rightarrow\ (y_1 + t^{Y,B}(y_1)\vec{e}, y_2 + t^{Y,B}(y_2)\vec{e}) \notin K. \tag{5.9}$$

With these properties in mind we will now give a recursive definition of $C^{Y,B}_k$ and $\tau^{Y,B}_k$ for a fixed $(Y,B) \in \mathcal{Y} \times \mathcal{E}$, where B is a subset of $E(Y)$. In the k^{th} construction step ($k \ge 0$) let

$$t^{Y,B}_k \ :=\ t^{Y,B}_0 \wedge \bigwedge_{0\le i<k} m_{C^{Y,B}_i,\tau^{Y,B}_i} \ =\ t^{Y,B}_{k-1} \wedge m_{C^{Y,B}_{k-1},\tau^{Y,B}_{k-1}},$$

$$\text{where} \quad t^{Y,B}_0 \ :=\ \tau_{R,n}(|.|) \quad \text{and} \quad m_{C^{Y,B}_i,\tau^{Y,B}_i} := \bigwedge_{y\in C^{Y,B}_i} m_{y,\tau^{Y,B}_i}.$$

The auxiliary functions $m_{y',t}$ will be defined later. Let $P^{Y,B}_k$ be the set of points of $Y \setminus (C^{Y,B}_0 \cup \ldots \cup C^{Y,B}_{k-1})$ at that $t^{Y,B}_k$ is minimal, and let $\tau^{Y,B}_k$ be the corresponding minimal value, so that $\tau^{Y,B}_k = t^{Y,B}_k(P^{Y,B}_k)$, i.e. $\tau^{Y,B}_k = t^{Y,B}_k(y)$ for all $y \in P^{Y,B}_k$. Let $C^{Y,B}_k$ be the B-cluster of the set $P^{Y,B}_k$ and $T^{Y,B}_k := id + t^{Y,B}_k\vec{e}$. The recursion stops when $Y \setminus (C^{Y,B}_0 \cup \ldots \cup C^{Y,B}_m) = \emptyset$, which occurs for a finite value of $m(Y,B) := m$. If it is clear from the context which configuration is

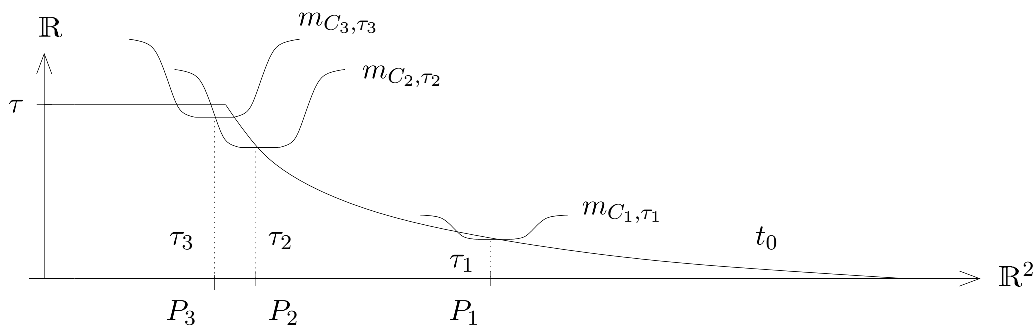

Figure 4: Construction of t_k in the case that $C_k = P_k$ is one point only.

considered, we may omit the dependence on Y and B in the above notations. If $Y_{\Lambda_n^c} \neq \emptyset$ then for every $y \in Y_{\Lambda_n^c}$ we have $\tau_{R,n}(|y|) = 0$ by (5.4), so $y \in P_0$ and $\tau_0 = 0$. This implies the second part of property (2). t_k is defined to be $t_0 = \tau_{R,n}(|.|)$ modified by local distortions $m_{y',t}$. On the one hand we have thus ensured that $t_k - t_0$ is small, i.e. $\tau_k \approx \tau_{R,n}(|y|)$ for all $y \in P_k$, which will give us hold on the density in property (4). On the other hand the auxiliary functions of the form $m_{y',t}$ slow down the translation locally near every point y' with known translation distance t, see Figure 4. This will ensure properties (5.8) and (5.9). For $y' \in \mathbb{R}^2_S$ and $t \in \mathbb{R}$ let the auxiliary function $m_{y',t} : \mathbb{R}^2_S \to \overline{\mathbb{R}}$ be given by

$$m_{y',t}(y) := \begin{cases} t & \text{if } h_{y',t} c_f > \frac{1}{2} \\ t + h_{y',t} f_K(y', y) + \infty\, 1_{\{f_K(y',y)=1\}} & \text{otherwise,} \end{cases}$$
$$\text{where} \quad h_{y',t} := |\tau_{R,n}(|y'| - c_K) - t|.$$

Note that the first case in the definition of $m_{y',t}$ has been introduced in order

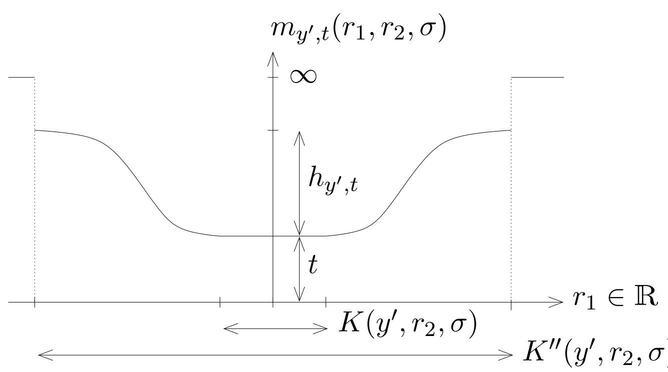

Figure 5: Graph of $m_{y',t}(., r_2, \sigma)$

to bound the slope of $m_{y',t}$. In Section 6.2 we show important properties of this auxiliary function, but for the moment we will content ourselves with the intuition given by Figure 5. Using Lemma 8 one can show that all above objects are measurable with respect to the considered σ-algebras. In the rest of this

section we will show that the above construction has indeed all the required properties.

Lemma 10 *The construction satisfies* (5.7), (5.8) *and* (5.9).

Lemma 11 *For good configurations* $(Y, B) \in G_{R,n}$ *we have*

$$(\mathfrak{T}^B_{R,n} Y - \tau\vec{e})_{\Lambda_{n'-1}} \;=\; Y_{\Lambda_{n'-1}} \qquad \textit{and} \qquad (\mathfrak{T}^B_{R,n} Y)_{\Lambda_n{}^c} \;=\; Y_{\Lambda_n{}^c}. \tag{5.10}$$

Lemma 12 *The transformation* $\mathfrak{T}_{R,n} : \mathcal{Y} \times \mathcal{E} \to \mathcal{Y} \times \mathcal{E}$ *is bijective.*

Actually in the proof of Lemma 12 we construct the inverse of $\mathfrak{T}_{R,n}$. This is needed in the proof of Lemma 13, where we will show for every $\bar{Y} \in \mathcal{Y}$ that $\nu_{\Lambda_n} \otimes \pi'_n(.|\bar{Y})$ is absolutely continuous with respect to $\nu_{\Lambda_n} \otimes \pi'_n(.|\bar{Y}) \circ \mathfrak{T}^{-1}_{R,n}$ with density $\varphi_{R,n} \circ \mathfrak{T}^{-1}_{R,n}$, where

$$\varphi_{R,n}(Y, B) \;:=\; \prod_{k=0}^{m(Y,B)} \; \prod_{y \in P_k^{Y,B}} \left|1 + \partial_{\vec{e}} t_k^{Y,B}(y)\right|. \tag{5.11}$$

Here $\partial_{\vec{e}}$ is the spatial derivative in direction $\vec{e}$. The proof will also show that definition (5.11) makes sense $\nu_{\Lambda_n} \otimes \pi'_n(\,.\,|\bar{Y})$-a.s., in that all derivatives exist. We note that for every $y \in Y_{\Lambda_n^c}$ the corresponding factor in the definition of $\varphi_{R,n}(Y, B)$ equals 1, so the above product in fact is a finite product.

Lemma 13 *For every* $\bar{Y} \in \mathcal{Y}$ *and every* $\mathcal{F}_{\mathcal{Y}} \otimes \mathcal{F}_{\mathcal{E}}$*-measurable function* $f \geq 0$

$$\int d\nu_{\Lambda_n} \otimes \pi'_n(.|\bar{Y})\, (f \circ \mathfrak{T}_{R,n} \cdot \varphi_{R,n}) \;=\; \int d\nu_{\Lambda_n} \otimes \pi'_n(.|\bar{Y})\, f. \tag{5.12}$$

Considering (3.5) we also need the backwards translation. So let $\bar{\mathfrak{T}}_{R,n}$, $\bar{\mathfrak{T}}^B_{R,n}$, $\bar{\mathfrak{T}}^Y_{R,n}$, and $\bar{\varphi}_{R,n}$ be defined analogously to the above objects, where now $\vec{e}$ is replaced by $-\vec{e}$. The previous lemmas apply analogously to this deformed backwards translation. We note that $\bar{\mathfrak{T}}_{R,n}$ is not the inverse of $\mathfrak{T}_{R,n}$.

5.5 Final steps of the proof

Let us now consider the $\delta > 0$ and the test set D chosen in Section 5.1. We identify D with $D \times \mathcal{E}$ and use the shorthand notation $D_{R,n} := D \cap G_{R,n}$. With a view to Lemma 1 we aim at showing that

$$\mu \otimes \pi_n(\mathfrak{T}_{R,n} D_{R,n}) + \mu \otimes \pi_n(\bar{\mathfrak{T}}_{R,n} D_{R,n}) - 2(1-\delta)\mu \otimes \pi_n(D_{R,n}) \tag{5.13}$$

is nonnegative. By (5.3) and Lemma 13 the first term of (5.13) equals

$$\int \mu(d\bar{Y}) \frac{1}{Z^{U,z}_{\Lambda_n}(\bar{Y})} \int \nu_{\Lambda_n} \otimes \pi'_n(dY, dB|\bar{Y})\; 1_{\mathfrak{T}_{R,n} D_{R,n}} \circ \mathfrak{T}_{R,n}(Y, B)$$
$$z^{\#(\mathfrak{T}^B_{R,n} Y)_{\Lambda_n}}\; \varphi_{R,n}(Y, B)\; e^{-H^{\bar{U}}_{\Lambda_n}(\mathfrak{T}^B_{R,n} Y)} \prod_{b \in \mathfrak{T}^Y_{R,n} B} (e^{u(b)} - 1).$$

By Lemma 12 $\mathfrak{T}_{R,n}$ is bijective, by (5.10) we have $\#(\mathfrak{T}^B_{R,n}Y)_{\Lambda_n} = \#Y_{\Lambda_n}$ and by construction any two particles connected by a bond are translated the same distance. Hence the above integrand simplifies to

$$1_{D_{R,n}}(Y,B)\, z^{\#Y_{\Lambda_n}}\, e^{\log\varphi_{R,n}(Y,B)-H^{\bar U}_{\Lambda_n}(\mathfrak{T}^B_{R,n}Y)} \prod_{b\in B}(e^{u(b)}-1).$$

Treating the other terms analogously, (5.13) can be seen to equal

$$\int \mu(d\bar Y)\frac{1}{Z^{U,z}_{\Lambda_n}(\bar Y)} \int \nu_{\Lambda_n}\otimes\pi'_n(dY,dB|\bar Y)\, 1_{D_{R,n}}(Y,B)\, z^{\#Y_{\Lambda_n}} \prod_{b\in B}(e^{u(b)}-1)$$
$$\times \left[e^{\log\varphi_{R,n}(Y,B)-H^{\bar U}_{\Lambda_n}(\mathfrak{T}^B_{R,n}Y)} + e^{\log\bar\varphi_{R,n}(Y,B)-H^{\bar U}_{\Lambda_n}(\bar{\mathfrak{T}}^B_{R,n}Y)} - 2(1-\delta)e^{-H^{\bar U}_{\Lambda_n}(Y)}\right].$$

The convexity of the exponential function implies that the sum of the first two terms in the last bracket is greater or equal to

$$2\, e^{\frac{1}{2}(\log\varphi_{R,n}(Y,B)+\log\bar\varphi_{R,n}(Y,B)-H^{\bar U}_{\Lambda_n}(\mathfrak{T}^B_{R,n}Y)-H^{\bar U}_{\Lambda_n}(\bar{\mathfrak{T}}^B_{R,n}Y))},$$

and here we can estimate the exponent using the following lemma:

Lemma 14 *For $(Y,B)\in G_{R,n}$ we have*

$$\log\varphi_{R,n}(Y,B)+\log\bar\varphi_{R,n}(Y,B)\ \ge\ -\delta \qquad \textit{and} \tag{5.14}$$

$$H^{\bar U}_{\Lambda_n}(\mathfrak{T}^B_{R,n}Y)+H^{\bar U}_{\Lambda_n}(\bar{\mathfrak{T}}^B_{R,n}Y)\ \le\ 2H^{\bar U}_{\Lambda_n}(Y)+\delta. \tag{5.15}$$

Using $e^{-\delta}\ge 1-\delta$, this establishes the nonnegativity of the above bracket and thus of (5.13). So we have shown

$$\mu\otimes\pi_n(\mathfrak{T}_{R,n}D_{R,n})\ +\ \mu\otimes\pi_n(\bar{\mathfrak{T}}_{R,n}D_{R,n})\ \ge\ 2\mu\otimes\pi_n(D_{R,n})-2\delta. \tag{5.16}$$

We would like to replace $D_{R,n}$ by D. Using $D\in\mathcal{F}_{\mathcal{Y},\Lambda_{n'-1}}$ and (5.10) we obtain

$$\forall\,(Y,B)\in D_{R,n}: \quad (\mathfrak{T}^B_{R,n}Y-\tau\vec e)_{\Lambda_{n'-1}}\in D, \quad \text{i.e. } \mathfrak{T}^B_{R,n}Y\in D+\tau\vec e,$$

and an analogous result for the backwards transformation. Hence

$$\mathfrak{T}_{R,n}(D_{R,n})\subset D+\tau\vec e \quad \text{and} \quad \bar{\mathfrak{T}}_{R,n}(D_{R,n})\subset D-\tau\vec e. \tag{5.17}$$

Lemma 15 *If the integers $n>R$ are chosen big enough, then $\mu\otimes\pi_n(G^c_{R,n})\le\delta$.*

For the proof of Theorem 2 we choose such $n>R$. Using (5.17) and Lemma 15 we deduce $\mu(D+\tau\vec e)+\mu(D-\tau\vec e)\ \ge\ 2\mu(D)-4\delta$ from (5.16). Taking the limit $\delta\to 0$, the claim of the theorem follows from Lemma 1.

6 Proof of the lemmas from Section 5

6.1 Convergence of energy sums: Lemma 9

Let $n \in \mathbb{N}$. For every $Y \in \mathcal{Y}$ we have

$$H^{\tilde{u}}_{\Lambda_n}(Y) = \sum_{b \in E_n(Y)} \tilde{u}(b) \leq \sum_{y_1, y_2 \in Y_{\Lambda_n}}^{\neq} \tilde{u}(y_1, y_2) + \sum_{y_1 \in Y_{\Lambda_n}} \sum_{y_2 \in Y_{\Lambda_n^c}} \tilde{u}(y_1, y_2),$$

and integrating this and applying Lemma 6 for $\nu_{\Lambda_n}(.|\bar{Y})$ and μ we obtain

$$\begin{aligned} \int \mu \otimes \nu_{\Lambda_n}(dY) H^{\tilde{u}}_{\Lambda_n}(Y) &\leq \int_{\Lambda_n} dy_1 \Big(\int_{\Lambda_n} dy_2 \tilde{u}(y_1, y_2) + z\xi \int_{\Lambda_n^c} dy_2 \tilde{u}(y_1, y_2) \Big) \\ &\leq \int_{\Lambda_n} dy_1 (1 + z\xi) c_\xi \quad \leq \quad 4n^2 (1 + z\xi) c_\xi \quad < \quad \infty, \end{aligned}$$

where we have estimated the integrals over y_2 by c_ξ using (5.1). Thus we have proved the first assertion. However, μ is absolutely continuous with respect to $\mu \otimes \nu_{\Lambda_n}$, which follows from (3.1) and the definition of the conditional Gibbs distribution. Hence the first assertion implies the second one.

6.2 Properties of the auxiliary function

A function $t : I \to \mathbb{R}$ on an interval I is called *hLd*, i.e. 1/2-Lipschitz-continuous and differentiable at all but at most countably many points, if $|t(r) - t(r')| \leq \frac{1}{2}|r - r'|$ for all $r, r' \in I$, and if there is a countable set $M \subset I$ such that f is differentiable in every point of $M \setminus I$. The following lemmas show why we consider this type of function:

Lemma 16 *Let $t : \mathbb{R} \to \mathbb{R}$ be hLd. Then the transformation $T : \mathbb{R} \to \mathbb{R}$, $T := id + t$, is bijective, strictly increasing, continuous, and differentiable a.e., and the Lebesgue transformation formula holds:*

$$\int g(T(r)) T'(r) dr = \int g(r') dr' \quad \text{for all measurable } g \geq 0. \tag{6.1}$$

Proof: We only need the 1/2-Lipschitz-continuity of t, which implies

$$\frac{1}{2}(r - r') \leq T(r) - T(r') \leq \frac{3}{2}(r - r') \quad \text{for all } r \geq r' \in I,$$

so T is bijective, strictly increasing, and Lipschitz-continuous. The inverse T^{-1} also is continuous and bijective, thus $\tilde{\lambda} := \lambda \circ T$ is a measure on $(\mathbb{R}, \mathcal{B})$. By the Lebesgue-Vitali differentiation theorem the Lipschitz-continuity of T implies that T is differentiable a.e. and $\frac{d\tilde{\lambda}}{d\lambda} = T'$. Thus the transformation theorem implies (6.1). $\square$

Lemma 17 *If $t_1, t_2 : I \to \mathbb{R}$ are hLd functions on an interval I, then so is $t := t_1 \wedge t_2$, and we have $t'(s) \in \{t_1'(s), t_2'(s)\}$ whenever $t'(s)$ exists.*

Proof: The 1/2-Lipschitz-continuity of t follows from the inequality

$$\forall\, a_i, b_i \in \mathbb{R}: \quad |a_1 \wedge a_2 - b_1 \wedge b_2| \leq |a_1 - b_1| \vee |a_2 - b_2|.$$

For the differentiability let $M_i \subset I$ be a countable set such that t_i is differentiable on $I \setminus M_i$. Furthermore let

$$M_3 := \{r \in I \setminus (M_1 \cup M_2) : t_1(r) = t_2(r), t_1'(r) \neq t_2'(r)\}.$$

It is easy to check that every point of M_3 is isolated, so M_3 is countable. But $t_1 \wedge t_2$ is differentiable on $I \setminus (M_1 \cup M_2 \cup M_3)$. Indeed, let $r \in I \setminus (M_1 \cup M_2)$. If $t_1(r) \neq t_2(r)$, then t coincides with one of the two functions in a neighborhood of r, and if $t_1(r) = t_2(r)$ and $t_1'(r) = t_2'(r)$, then t is differentiable in r with $t'(r) = t_1'(r) = t_2'(r)$. □

Let us call a function $t : \mathbb{R}^2_S \to \mathbb{R}$ $\vec{e}$-1/2-Lipschitz-continuous, $\vec{e}$-differentiable, or $\vec{e}$-hLd if for all $r_2 \in \mathbb{R}, \sigma \in S$ the function $t(., r_2, \sigma)$ has the corresponding property.

Lemma 18 *For all $y' \in \mathbb{R}^2_S$ and $t \in \mathbb{R}$ $\tau_n(|.|) \wedge m_{y',t}$ is $\vec{e}$-hLd.*

Proof: Let $y' \in \mathbb{R}^2_S$ and $t \in \mathbb{R}$. The claimed properties concern the first spatial component only, so for fixed $r_2 \in \mathbb{R}$ and $\sigma \in S$ we consider the functions $\tilde{\tau} := \tau_n(|(., r_2, \sigma)|)$, $\tilde{f} := f_K(y', ., r_2, \sigma)$, $\tilde{m}^f := t + h_{y',t}\tilde{f}$, $\tilde{m} := \tilde{m}^f + \infty 1_{\{\tilde{f}=1\}}$. It suffices to show that $\tilde{\tau} \wedge t$ is hLd for $h_{y',t}c_f > 1/2$, and $\tilde{\tau} \wedge \tilde{m}$ is hLd for $h_{y',t}c_f \leq 1/2$. In order to get rid of the infinite part of $\tilde{m}$ in the second case we define I to be the convex hull of the closure of $\{\tilde{f} < 1\}$. I is a bounded closed interval, and we claim that

$$\tilde{\tau} \wedge \tilde{m} = \tilde{\tau} \ \text{ on } \ \overline{\mathbb{R} \setminus I} \quad \text{ and } \quad \tilde{\tau} \wedge \tilde{m} = \tilde{\tau} \wedge \tilde{m}^f \ \text{ on } \ I. \tag{6.2}$$

Provided this is true, we are done by Lemma 17 as $\tilde{m}^f$ is hLd for $h_{y',t}c_f \leq 1/2$ (by definition of c_f) and $\tilde{\tau}$ is hLd ($\tilde{\tau}$ is Lipschitz-continuous with Lipschitz-constant $\tau/Q(n-R) \leq 1/2$, where we have used $n - R \geq 1$ and $\tau \leq 1/2$). For a proof of (6.2) we first observe that we have $\tilde{f} = 1$ on $\overline{\mathbb{R} \setminus I}$ by the continuity of $\tilde{f}$ and thus $\tilde{m} = \infty$, which gives the first claim. For the second claim it suffices to show that for all $r \in I$ with $\tilde{f}(r) = 1$ we have $\tilde{m}^f(r) \geq \tilde{\tau}(r)$. So let $r \in I$ with $\tilde{f}(r) = 1$. We observe that I is contained in the convex hull of the closure of $K''(y', r_2, \sigma)$, as $\{\tilde{f} < 1\} \subset K''(y', r_2, \sigma)$ by definition of $\tilde{f}$. Thus for $y := (r, r_2, \sigma)$ we have $|y' - y| \leq c_K$, which implies $|y'| - c_K \leq |y|$. As $\tau_{R,n}$ is decreasing we obtain

$$\tilde{\tau}(r) \;=\; \tau_{R,n}(|y|) \;\leq\; \tau_{R,n}(|y'| - c_K) \;\leq\; t + h_{y',t} \;=\; \tilde{m}^f(r)$$

by choice of $h_{y',t}$, and we are done. □

6.3 Properties of the construction: Lemma 10

$t_k^{Y,B}$ is the minimum of finitely many functions of the form $\tau_n(|.|) \wedge m_{y',t}$, where $y' \in \mathbb{R}^2_S$ and $t \in \mathbb{R}$. So the preceeding lemmas imply the following monotonicity and regularity properties of $t_k^{Y,B}$ and $T_k^{Y,B}$:

Lemma 19 *For $Y \in \mathcal{Y}$, $B \subset E(Y)$ and $k \geq 0$ we have that $t_k^{Y,B}$ is $\vec{e}$-hLd and for every $y \in \mathbb{R}^2_S$ $\partial_{\vec{e}} t_k^{Y,B}(y)$ equals 0 or $\partial_{\vec{e}} t_0^{Y,B}(y)$ or $\partial_{\vec{e}} m_{y', t^{Y,B}(y')}(y)$ for some $y' \in Y$ with $(y, y') \in K''$. Furthermore $T_k^{Y,B}$ is $\leq_{\vec{e}}$-increasing, $\vec{e}$-continuous, bijective, and $T_k^{Y,B}$ as a function of the first spatial coordinate satisfies* (6.1).

In the proofs of many of the following lemmas we need a way to calculate the translation distance of an arbitrary particle $y \in C_k^{Y,B}$ without knowing $P_k^{Y,B}$. This can be done using the following easy fact:

Lemma 20 *For $Y \in \mathcal{Y}$, $B \subset E(Y)$ and $k \geq 0$ we have*

$$\tau_k^{Y,B} = t_{k+1}^{Y,B}(y) \quad \text{for all } y \in C_k^{Y,B}. \tag{6.3}$$

Proof: For $y \in C_k$ we have $t_{k+1}(y) = t_k(y) \wedge \bigwedge_{y' \in C_k} m_{y', \tau_k}(y) = \tau_k$. Here we have used $t_k(y) \geq \tau_k$, which holds by definition of τ_k, $m_{y', \tau_k}(y) \geq \tau_k$, and $m_{y, \tau_k}(y) = \tau_k$. □

For (5.7) it suffices to observe that for every $1 \leq k \leq m$ and $y \in P_k$ we have

$$\tau_k = t_k(y) = t_{k-1}(y) \wedge \bigwedge_{y' \in C_{k-1}} m_{y', \tau_{k-1}}(y) \geq \tau_{k-1}.$$

This follows from the definition of τ_k and t_k, from $t_{k-1}(y) \geq \tau_{k-1}$ by the definition of τ_{k-1}, and from $m_{y',t} \geq t$. Now we will show (5.8) and

$$s \in [-1, 1], y, y' \in Y, (y, y') \notin K \;\Rightarrow\; (y, y' + s(t^{Y,B}(y') - t^{Y,B}(y))\vec{e}) \notin K. \tag{6.4}$$

By the $\vec{e}$-invariance of K (5.9) is equivalent to the special case $s = 1$ of (6.4). Let $y, y' \in Y$ and $s \in [-1, 1]$. Without loss of generality we may suppose that $y = y_i \in C_i$ and $y' = y_j \in C_j$, where $0 \leq i < j$. (For $i = j$ we have $t^{Y,B}(y_i) = t^{Y,B}(y_j)$, so (5.8) and (6.4) are obvious.) We now observe that $y_j \in \Lambda^i := \{y \in \mathbb{R}^2_S : t_i(y) \geq \tau_i\}$ and

$$\forall\, y \in K(y_i) \cap \Lambda^i : \;\; t_{j+1}(y) = t_i(y) \wedge \bigwedge_{i \leq k \leq j} m_{C_k, \tau_k}(y) = \tau_i. \tag{6.5}$$

This holds as $t_i(y) \geq \tau_i$ by definition of Λ^i, $m_{C_k,\tau_k} \geq \tau_i$ by (5.7), and $m_{C_i,\tau_i}(y) = \tau_i$ by $y \in K(y_i)$. If $(y_i, y_j) \in K$, then $y_j \in K(y_i) \cap \Lambda^i$, so (6.5) and (6.3) imply $\tau_j = t_{j+1}(y_j) = \tau_i$, which shows (5.8). For (6.4) suppose $(y_i, y_j) \notin K$ and let $T^s_{j+1} := id + s \cdot t_{j+1}\vec{e}$. We have $y_j \in \Lambda^i \setminus K(y_i)$ and $\tau_j = t_{j+1}(y_j)$ by (6.3), so it suffices to show that

$$T^s_{j+1}(\Lambda^i \setminus K(y_i)) = \Lambda^i \setminus K(y_i) + s\tau_i\vec{e}, \tag{6.6}$$

as this implies $y_j + s\tau_j\vec{e} \notin K(y_i) + s\tau_i\vec{e}$. In order to show (6.6) we fix $\sigma \in S$ and $r \in \mathbb{R}$. Continuity of $t_i(., r, \sigma)$ implies $t_i(., r, \sigma) = \tau_i$ on $\partial\Lambda^i(., r, \sigma)$. Just as in the proof (6.5) it follows that $t_{j+1}(., r, \sigma) = \tau_i$ on $\partial\Lambda^i$. But $T^s_{j+1}(., r, \sigma)$ is increasing, continuous, and bijective, which can be shown as in the proof of Lemma 19. So $T^s_{j+1}(\Lambda^i) = \Lambda^i + s\tau_i\vec{e}$, and combining this with (6.5) we are done.

6.4 Properties of the deformed translation: Lemma 11

The following lemma shows how to estimate translation distances of particles.

Lemma 21 *For good configurations $(Y,B) \in G_{R,n}$ we have*

$$\forall\, y \in Y: \quad 0 \,\le\, \tau_{R,n}^{\wedge,Y,B_+}(y) \,\le\, t^{Y,B}(y) \,\le\, \tau_{R,n}(|y|) \,\le\, \tau. \tag{6.7}$$

Proof: The first and fourth inequality are a consequence of (5.4), and for the third it suffices to observe that for $y \in C_k$ we have $\tau_k \le t_k(y) \le t_0(y)$ by the definition of τ_k. For the second inequality we define

$$\Sigma_1(R,n,Y,B) \,:=\, \sum_{y,y' \in Y} 1_{\{y \overset{Y,B_+}{\longleftrightarrow} y'\}} \tau_{R,n}^q(y,y') 4c_f^2, \tag{6.8}$$

$$\text{where} \quad \tau_{R,n}^q(y,y') \,:=\, 1_{\{|y| \le |y'|\}} |\tau_{R,n}(|y|-c_K) - \tau_{R,n}(|y'|)|^2. \tag{6.9}$$

We have $\Sigma_1(R,n,Y,B) < 1$ by $(Y,B) \in G_{R,n}$ and by definition of the set $G_{R,n}$ in (5.5). Hence every summand of Σ_1 is < 1, and if we choose y' to be a particle in $C_{Y,B_+}(y)$ such that $\tau_{R,n}(|.|)$ is minimal and $|y'| \ge |y|$ this implies

$$\forall y \in Y:\; 2c_f \big|\tau_{R,n}(|y|-c_K) - \tau_{R,n}^{\wedge,Y,B_+}(y)\big| \,\le\, 1. \tag{6.10}$$

We will use this to show that all distortion functions $m_{y',t}$ in the definition of $t^{Y,B}(y)$ only have local influence in that in the definition of $m_{y',t}$ we have the second case ($h_{y',t}c_f \le 1/2$), which is needed in the following proof of

$$\tau_{R,n}^{\wedge,Y,B_+}(y) \,\le\, \tau_k^{Y,B} \quad \text{for all } y \in C_k^{Y,B}$$

by induction on k. For $k=0$ we have equality. For the inductive step $k-1 \to k$ let $i \le k-1$. By the third inequality of (6.7), the inductive hypothesis, and (6.10) we observe that for all $y_i \in C_i$ we have

$$0 \,\le\, \big(\tau_{R,n}(|y_i|-c_K) - \tau_i\big)c_f \,\le\, \big(\tau_{R,n}(|y_i|-c_K) - \tau_{R,n}^{\wedge,Y,B_+}(y_i)\big)c_f \,\le\, 1/2,$$

so $h_{y,\tau_i}c_f \le 1/2$. Therefore m_{y_i,τ_i} is local in that $m_{y_i,\tau_i}(y) = \infty$ for all $y \in P_k$ such that $(y_i,y) \notin K''$. Thus

$$\tau_k \,=\, t_k(y) \,=\, t_0(y) \wedge \bigwedge_{y_i \in C_i: i<k, (y_i,y) \in K''} m_{y_i,\tau_i}(y) \,\ge\, \tau_{R,n}^{\wedge,Y,B_+}(y),$$

where the last step follows from $m_{y_i,\tau_i}(y) \ge \tau_i \ge \tau_{R,n}^{\wedge,Y,B_+}(y_i)$, which is due to the induction hypothesis, and from $y_i \overset{Y,B_+}{\longleftrightarrow} y$ for $(y_i,y) \in K''$. $\square$

We note that the proof Lemma 21 also shows that in the construction of $\mathfrak{T}_{R,n}(Y,B)$ for a good configuration $(Y,B) \in G_{R,n}$ all appearing distortion functions $m_{y',t}$ only have local influence as $h_{y',t}c_f \le 1/2$. Now we will prove Lemma 11. It suffices to show for all $(Y,B) \in G_{R,n}$ and $y \in Y$ that

$$\begin{array}{llll} y \in \Lambda_{n'} \;\Rightarrow\; t^{Y,B}(y) = \tau, & & y \in \Lambda_{n'}^c \;\Rightarrow\; y + t^{Y,B}(y)\vec{e} - \tau\vec{e} \notin \Lambda_{n'-1} \\ y \in \Lambda_n^c \;\Rightarrow\; t^{Y,B}(y) = 0, & \text{and} & y \in \Lambda_n \;\Rightarrow\; y + t^{Y,B}(y)\vec{e} \in \Lambda_n. \end{array} \tag{6.11}$$

So let $(Y,B) \in G_{R,n}$ and $y \in Y$. The first assertion of (6.11) now follows from (5.6) and (6.7). The second assertion is an immediate consequence of $0 \le \tau - t^{Y,B}(y) \le 1$, which follows from (6.7) and $\tau \le 1$. The third assertion follows from (6.7) and (5.4), and for the fourth assertion let $y \in \Lambda_n$. As

$$y \le_{\vec{e}} y + t^{Y,B}(y)\vec{e} \le_{\vec{e}} T_0^{Y,B}(y)$$

by (6.7), it suffices to show that also $T_0^{Y,B}(y) \in \Lambda_n$. This however follows from $T_0^{Y,B} = id$ on ${\Lambda_n}^c$ and the bijectivity of $T_0^{Y,B}$ from Lemma 19.

6.5 Bijectivity of the transformation: Lemma 12

We construct the inverse transformation $\tilde{\mathfrak{T}}_{R,n}$ recursively, similarly to the construction of $\mathfrak{T}_{R,n}$, i.e. from a given configuration $(\tilde{Y},\tilde{B})$ we will choose sets of points $\tilde{C}_k^{\tilde{Y},\tilde{B}}$ and translate them by $\tilde{\tau}_k^{\tilde{Y},\tilde{B}}$ in direction $-\vec{e}$. To get an idea how to define the inverse transformation we start with a fixed configuration $Y \in \mathcal{Y}$, $B \subset E(Y)$ and set $(\tilde{Y},\tilde{B}) := \mathfrak{T}_{R,n}(Y,B)$. In the construction of $\mathfrak{T}_{R,n}(Y,B)$ we defined a partition of Y into sets of particles C_k, corresponding sets P_k, and translation distances τ_k. We denote the corresponding image sets by $\tilde{P}_k := P_k + \tau_k\vec{e}$ and $\tilde{C}_k := C_k + \tau_k\vec{e}$, see Figure 6. For the construction

$\mathfrak{T}_{R,n} : (Y,B) \mapsto (\tilde{Y},\tilde{B})$ $\qquad$ $\tilde{\mathfrak{T}}_{R,n} : (\tilde{Y},\tilde{B}) \mapsto (Y,B)$

Figure 6: Construction of the inverse $\tilde{\mathfrak{T}}_{R,n}$ of $\mathfrak{T}_{R,n}$.

of the inverse transformation we have to find a method to identify the sets $\tilde{C}_k$ among the points of $\tilde{Y}$ without knowing (Y,B). Suppose now that we have already found $\tilde{C}_i$, $\tilde{P}_i$ and τ_i for all $i < k$, then with this information we are able to reconstruct $C_i := \tilde{C}_i - \tau_i\vec{e}$ and thus t_k and T_k. The following lemma tells us, how to find $\tilde{P}_k$ in that case, which will also give us $\tilde{C}_k$, $P_k := T_k^{-1}(\tilde{P}_k)$ and $\tau_k := t_k(P_k)$.

Lemma 22 *Let $0 \le k \le m$. $\tilde{P}_k$ is the set of points of $\tilde{Y} \setminus \bigcup_{i \le k-1} \tilde{C}_i$ where $t_k \circ T_k^{-1}$ attains its minimum value.*

Proof: We first observe that for all k by definition of t_k we have

$$T_k^{-1} + t_k \circ T_k^{-1}\vec{e} = id. \tag{6.12}$$

Since $t_{k+1} \le t_k$, we also have $T_{k+1} \le_{\vec{e}} T_k$, and therefore $T_k^{-1} \le_{\vec{e}} T_{k+1}^{-1}$ by the $\vec{e}$-monotonicity of T_{k+1}^{-1} from Lemma 19. Together with (6.12) this implies

$$t_{k+1} \circ T_{k+1}^{-1} \le t_k \circ T_k^{-1}. \tag{6.13}$$

Now let $0 \le k \le m$, $\tilde{y}_k \in \tilde{P}_k$, and $\tilde{y}_l \in \tilde{C}_l$ for some $l \ge k$. Then $y_k := \tilde{y}_k - \tau_k \vec{e} \in P_k$ and $y_l := \tilde{y}_l - \tau_l \vec{e} \in C_l$. By definition and by (6.3) we have $T_k(y_k) = \tilde{y}_k$, $t_{l+1}(y_l) = \tau_l$, and $T_{l+1}(y_l) = \tilde{y}_l$. Using (5.7) and (6.13) we deduce

$$t_k(T_k^{-1}(\tilde{y}_k)) \,=\, \tau_k \,\le\, \tau_l \,=\, t_{l+1}(T_{l+1}^{-1}(\tilde{y}_l)) \,\le\, t_k(T_k^{-1}(\tilde{y}_l)).$$

If for the given $\tilde{y}_l$ we have equality, all inequalities in the previous line have to be equalities, so $\tau_k = \tau_l$ and $t_{l+1}(T_{l+1}^{-1}(\tilde{y}_l)) = t_k(T_k^{-1}(\tilde{y}_l))$. Combining this with (6.12) we get $y_l = T_{l+1}^{-1}(\tilde{y}_l) = T_k^{-1}(\tilde{y}_l)$, so $T_k(y_l) = \tilde{y}_l$ and thus $t_k(y_l) = \tau_l = \tau_k$. By definition of P_k we conclude $y_l \in P_k$, so $\tilde{y}_l \in \tilde{P}_k$ and we are done. $\square$

Lemma 22 tells us exactly how to construct the inverse of $\mathfrak{T}_{R,n}$ recursively. So let $\tilde{Y} \in \mathfrak{Y}$ and $\tilde{B} \subset E(\tilde{Y})$. In the k^{th} construction step ($k \ge 0$) we define

$$\tilde{t}_k^{\tilde{Y},\tilde{B}} := \tilde{t}_{k-1}^{\tilde{Y},\tilde{B}} \wedge \bigwedge_{y \in \tilde{C}_{k-1}^{\tilde{Y},\tilde{B}} - \tilde{\tau}_{k-1}^{\tilde{Y},\tilde{B}}} m_{y,\tilde{\tau}_{k-1}^{\tilde{Y},\tilde{B}}}, \qquad \text{where} \quad \tilde{t}_0^{\tilde{Y},\tilde{B}} := \tau_{R,n}(|.|).$$

Let $\tilde{T}_k^{\tilde{Y},\tilde{B}} = id + \tilde{t}_k^{\tilde{Y},\tilde{B}} \vec{e}$, and define $\tilde{P}_k^{\tilde{Y},\tilde{B}}$ to be the set of particles of $\tilde{Y} \setminus (\tilde{C}_0^{\tilde{Y},\tilde{B}} \cup \ldots \cup \tilde{C}_{k-1}^{\tilde{Y},\tilde{B}})$ at that the minimum of $\tilde{t}_k^{\tilde{Y},\tilde{B}} \circ (\tilde{T}_k^{\tilde{Y},\tilde{B}})^{-1}$ is attained. Let $\tilde{\tau}_k^{\tilde{Y},\tilde{B}} := \tilde{t}_k^{\tilde{Y},\tilde{B}} \circ (\tilde{T}_k^{\tilde{Y},\tilde{B}})^{-1}(\tilde{P}_k^{\tilde{Y},\tilde{B}})$ be the corresponding minimal value and $\tilde{C}_k^{\tilde{Y},\tilde{B}}$ be the $\tilde{B}$-cluster of the set $\tilde{P}_k^{\tilde{Y},\tilde{B}}$. The recursion stops, when $\tilde{Y} \setminus (\tilde{C}_0^{\tilde{Y},\tilde{B}} \cup \ldots \cup \tilde{C}_{\tilde{m}}^{\tilde{Y},\tilde{B}}) = \emptyset$, which will occur for a finite value of $\tilde{m}(\tilde{Y},\tilde{B}) := \tilde{m}$. Again, sometimes we will omit the dependence on $\tilde{Y}$ and $\tilde{B}$ in our notations if it is clear from the context which configuration is considered. We need to show that the above construction is well defined, i.e. that $\tilde{T}_k^{\tilde{Y},\tilde{B}}$ is invertible in every step. Furthermore we need some more properties of the construction:

Lemma 23 *Let $\tilde{Y} \in \mathfrak{Y}$, $\tilde{B} \subset E(\tilde{Y})$ and $k \ge 0$. Then*

$$\tilde{t}_k^{\tilde{Y},\tilde{B}} \textit{ is } \vec{e}\textit{-hLd}, \quad \tilde{T}_k^{\tilde{Y},\tilde{B}} \textit{ is bijective and } \le_{\vec{e}}\textit{-increasing}, \tag{6.14}$$

$$(\tilde{T}_k^{\tilde{Y},\tilde{B}})^{-1} + \tilde{t}_k^{\tilde{Y},\tilde{B}} \circ (\tilde{T}_k^{\tilde{Y},\tilde{B}})^{-1} \vec{e} \,=\, id, \tag{6.15}$$

$$\forall\, c \in \mathbb{R}, y \in \mathbb{R}_S^2 :\, \tilde{t}_k^{\tilde{Y},\tilde{B}} \circ (\tilde{T}_k^{\tilde{Y},\tilde{B}})^{-1}(y) \,\ge\, c \,\Leftrightarrow\, \tilde{t}_k^{\tilde{Y},\tilde{B}}(y - c\vec{e}) \,\ge\, c, \tag{6.16}$$

$$\tilde{t}_k^{\tilde{Y},\tilde{B}} \,\le\, \tilde{t}_{k-1}^{\tilde{Y},\tilde{B}} \quad \textit{and} \quad \tilde{\tau}_{k-1}^{\tilde{Y},\tilde{B}} \,\le\, \tilde{\tau}_k^{\tilde{Y},\tilde{B}}, \tag{6.17}$$

$$\forall\, y \in \tilde{C}_k^{\tilde{Y},\tilde{B}} :\, \tilde{t}_{k+1}^{\tilde{Y},\tilde{B}} \circ (\tilde{T}_{k+1}^{\tilde{Y},\tilde{B}})^{-1}(y) \,=\, \tilde{\tau}_k^{\tilde{Y},\tilde{B}}. \tag{6.18}$$

Proof: The definitions of $\tilde{t}_k$ and $\tilde{T}_k$ are similar to those of t_k and T_k, so we can show (6.14) and (6.15) just as the corresponding properties in Lemma 19 and (6.12). For (6.16) we note that for $c \in \mathbb{R}$ and $y \in \mathbb{R}_S^2$ the equivalence

$$\begin{aligned} \tilde{t}_k \circ (\tilde{T}_k)^{-1}(y) \,\ge\, c \quad &\Leftrightarrow \quad (\tilde{T}_k)^{-1}(y) \,\le_{\vec{e}}\, y - c\vec{e} \\ &\Leftrightarrow \quad y \,\le_{\vec{e}}\, \tilde{T}_k(y - c\vec{e}) = y - c\vec{e} + \tilde{t}_k(y - c\vec{e})\vec{e} \end{aligned}$$

follows from (6.15) and (6.14). The first part of (6.17) is obvious and for the second part we observe that for $\tilde{y}_k \in \tilde{P}_k$ we have

$$\begin{aligned} &\tilde{t}_{k-1} \circ (\tilde{T}_{k-1})^{-1}(\tilde{y}_k) \ge \tilde{\tau}_{k-1} \quad \Rightarrow \quad \tilde{t}_{k-1}(\tilde{y}_k - \tilde{\tau}_{k-1}\vec{e}) \ge \tilde{\tau}_{k-1} \\ \Rightarrow \quad &\tilde{t}_k(\tilde{y}_k - \tilde{\tau}_{k-1}\vec{e}) \ge \tilde{\tau}_{k-1} \quad \Rightarrow \quad \tilde{\tau}_k = \tilde{t}_k \circ (\tilde{T}_k)^{-1}(\tilde{y}_k) \ge \tilde{\tau}_{k-1}, \end{aligned}$$

where the first statement holds by definition of $\tilde{P}_{k-1}$, the first and the third implication hold by (6.16), and the second implication is by definition of $\tilde{t}_k$. For (6.18) let $\tilde{y}_k \in \tilde{C}_k$. We have

$$\begin{aligned} \tilde{t}_k \circ \tilde{T}_k^{-1}(\tilde{y}_k) \,\geq\, \tilde{\tau}_k \quad &\Rightarrow \quad \tilde{t}_k(\tilde{y}_k - \tilde{\tau}_k \vec{e}) \,\geq\, \tilde{\tau}_k \\ \Rightarrow \quad \tilde{t}_{k+1}(\tilde{y}_k - \tilde{\tau}_k \vec{e}) \,=\, \tilde{\tau}_k \quad &\Rightarrow \quad \tilde{t}_{k+1} \circ \tilde{T}_{k+1}^{-1}(\tilde{y}_k) \,=\, \tilde{\tau}_k, \end{aligned}$$

where the first statement holds by definition, and the implications follow from (6.16), $\tilde{y}_k - \tilde{\tau}_k \vec{e} \in \tilde{C}_k - \tilde{\tau}_k \vec{e}$ and (6.15) respectively. □

For every $0 \leq k \leq \tilde{m}(\tilde{Y}, \tilde{B})$ and $\tilde{y}_k \in \tilde{C}_k^{\tilde{Y},\tilde{B}}$ let $\tilde{t}^{\tilde{Y},\tilde{B}}(\tilde{y}_k) := \tilde{\tau}_k^{\tilde{Y},\tilde{B}}$. This defines a translation distance function $\tilde{t}^{\tilde{Y},\tilde{B}} : \tilde{Y} \to \mathbb{R}$. We define

$$\begin{aligned} \tilde{\mathfrak{T}}_{R,n}^{\tilde{B}}(\tilde{Y}) \,&:=\, \bigcup_{k=0}^{\tilde{m}(\tilde{Y},\tilde{B})} (\tilde{C}_k^{\tilde{Y},\tilde{B}} - \tilde{\tau}_k^{\tilde{Y},\tilde{B}} \vec{e}) = \{y - \tilde{t}^{\tilde{Y},\tilde{B}}(y)\vec{e} : y \in \tilde{Y}\} \quad \text{and} \\ \tilde{\mathfrak{T}}_{R,n}^{\tilde{Y}}(\tilde{B}) \,&:=\, \{(y - \tilde{t}^{\tilde{Y},\tilde{B}}(y)\vec{e})(y' - \tilde{t}^{\tilde{Y},\tilde{B}}(y')\vec{e}) : yy' \in \tilde{B}\}. \end{aligned}$$

Now if $\tilde{B}$ is a not a subset of $E(\tilde{Y})$ we define $\tilde{\mathfrak{T}}_{R,n}^{\tilde{B}} = id$ and $\tilde{\mathfrak{T}}_{R,n}^{\tilde{Y}} = id$. Let

$$\tilde{\mathfrak{T}}_{R,n} : \mathfrak{Y} \times \mathcal{E} \to \mathfrak{Y} \times \mathcal{E}, \quad \tilde{\mathfrak{T}}_{R,n}(\tilde{Y}, \tilde{B}) \,:=\, (\tilde{\mathfrak{T}}_{R,n}^{\tilde{B}}(\tilde{Y}), \tilde{\mathfrak{T}}_{R,n}^{\tilde{Y}}(\tilde{B})).$$

By Lemma 8 we see again that all above objects are measurable with respect to the considered σ-algebras. The only difficulty is to show that $(\tilde{T}_k^{\tilde{Y},\tilde{B}})^{-1}(y)$ is measurable. This however follows from the $\vec{e}$-monotonicity of $\tilde{T}_k^{\tilde{Y},\tilde{B}}$. In order to show that $\tilde{\mathfrak{T}}_{R,n}$ really is the inverse of $\mathfrak{T}_{R,n}$ we need an analogue of Lemma 22. Let $\tilde{Y} \in \mathfrak{Y}$ and $\tilde{B} \subset E(\tilde{Y})$. Let $\tilde{t}_k$, $\tilde{T}_k$, $\tilde{C}_k$, $\tilde{P}_k$, and $\tilde{\tau}_k$ $(0 \leq k \leq \tilde{m})$ as above and denote $(Y, B) := \tilde{\mathfrak{T}}_{R,n}(\tilde{Y}, \tilde{B})$, $P_k := \tilde{P}_k - \tilde{\tau}_k \vec{e}$, and $C_k := \tilde{C}_k - \tilde{\tau}_k \vec{e}$, see Figure 6.

Lemma 24 *Let $0 \leq k \leq \tilde{m}$. P_k is the set of points of $Y \setminus \bigcup_{i \leq k-1} C_i$ where $\tilde{t}_k$ attains its minimal value.*

Proof: Let $0 \leq k \leq \tilde{m}$, $y_k \in P_k$, and $y_l \in C_l$ for some $l \geq k$. Then $\tilde{y}_l := y_l + \tilde{\tau}_l \vec{e} \in \tilde{C}_l$ and $\tilde{y}_k := y_k + \tilde{\tau}_k \vec{e} \in \tilde{P}_k$. By definition of $\tilde{\tau}_k$ and $\tilde{\tau}_l$, by (6.18) and (6.15) we have $\tilde{T}_k^{-1}(\tilde{y}_k) = y_k$, $\tilde{t}_{l+1}(\tilde{T}_{l+1}^{-1}(\tilde{y}_l)) = \tilde{\tau}_l$, and $\tilde{T}_{l+1}^{-1}(\tilde{y}_l) = y_l$. Thus from (6.17) we deduce

$$\tilde{t}_k(y_k) \,=\, \tilde{\tau}_k \,\leq\, \tilde{\tau}_l \,=\, \tilde{t}_{l+1}(\tilde{T}_{l+1}^{-1}(\tilde{y}_l)) \,=\, \tilde{t}_{l+1}(y_l) \,\leq\, \tilde{t}_k(y_l).$$

If for the given y_l we have equality, all inequalities in the previous line have to be equalities, so $\tilde{\tau}_k = \tilde{\tau}_l$ and $\tilde{t}_k(y_l) = \tilde{\tau}_l$, i.e. $\tilde{T}_k(y_l) = y_l + \tilde{\tau}_l \vec{e} = \tilde{y}_l$. This gives $\tilde{\tau}_k = \tilde{\tau}_l = \tilde{t}_k(y_l) = \tilde{t}_k(\tilde{T}_k^{-1}(\tilde{y}_l))$. By definition of $\tilde{P}_k$ we conclude $\tilde{y}_l \in \tilde{P}_k$, hence $y_l \in P_k$ and we are done. □

Lemma 25 *On $\mathfrak{Y} \times \mathcal{E}$ we have* $\quad \tilde{\mathfrak{T}}_{R,n} \circ \mathfrak{T}_{R,n} \,=\, id \quad$ *and* $\quad \mathfrak{T}_{R,n} \circ \tilde{\mathfrak{T}}_{R,n} \,=\, id.$

Proof: For the first part let $Y \in \mathcal{Y}$, $B \in \mathcal{E}$, and $(\tilde{Y}, \tilde{B}) := \mathfrak{T}_{R,n}(Y,B)$. If B is not a subset of $E(Y)$ we have $\tilde{\mathfrak{T}}_{R,n} \circ \mathfrak{T}_{R,n}(Y,B) = \tilde{\mathfrak{T}}_{R,n}(Y,B) = (Y,B)$ and we are done. Else it suffices to prove

$$\begin{aligned}
\tilde{t}_k^{\tilde{Y},\tilde{B}} &= t_k^{Y,B}, \quad \tilde{T}_k^{\tilde{Y},\tilde{B}} = T_k^{Y,B}, \quad \tilde{\tau}_k^{\tilde{Y},\tilde{B}} = \tau_k^{Y,B}, \\
\tilde{P}_k^{\tilde{Y},\tilde{B}} &= P_k^{Y,B} + \tau_k^{Y,B}\vec{e}, \quad \text{and} \quad \tilde{C}_k^{\tilde{Y},\tilde{B}} = C_k^{Y,B} + \tau_k^{Y,B}\vec{e}
\end{aligned} \tag{6.19}$$

for every $k \geq 0$ by induction on k. For the inductive step $k-1 \to k$ we observe that $\tilde{t}_k^{\tilde{Y},\tilde{B}} = t_k^{Y,B}$ by induction hypothesis, and $\tilde{T}_k^{\tilde{Y},\tilde{B}} = T_k^{Y,B}$ is an immediate consequence. Combining this with Lemma 22 and the definition of $\tilde{P}_k^{\tilde{Y},\tilde{B}}$ we get $\tilde{P}_k^{\tilde{Y},\tilde{B}} = P_k^{Y,B} + \tau_k^{Y,B}\vec{e}$ and $\tilde{\tau}_k^{\tilde{Y},\tilde{B}} = \tau_k^{Y,B}$. $\tilde{C}_k^{\tilde{Y},\tilde{B}} = C_k^{Y,B} + \tau_k^{Y,B}\vec{e}$ is an immediate consequence. The case $k=0$ can be shown similarly: Here $\tilde{t}_0^{\tilde{Y},\tilde{B}} = t_0^{Y,B}$ holds by definition and the rest again follows from Lemma 22.
For the second part let $\tilde{Y} \in \mathcal{Y}$, $\tilde{B} \in \mathcal{E}$, and $(Y,B) := \tilde{\mathfrak{T}}_{R,n}(\tilde{Y},\tilde{B})$. As above we may assume $\tilde{B} \subset E(\tilde{Y})$, and it suffices to show (6.19) by induction on k. Here the inductive step follows from Lemma 24. $\square$

6.6 Density of the transformed process: Lemma 13

By definition the left hand side of (5.12) equals

$$e^{-4n^2} \sum_{k \geq 0} \frac{1}{k!} I(k), \qquad \text{where } I(k) = \int_{{\Lambda_n}^k} dy \sum_{B \subset E_n(\bar{Y}_y)}{}' (f \circ \mathfrak{T}_{R,n} \cdot \varphi_{R,n})(\bar{Y}_y, B),$$

where the summation symbol $\sum'$ indicates that the sum extends over finite subsets only, and we have used the shorthand notation $\bar{Y}_y = \{y_1, \ldots, y_k\} \cup \bar{Y}_{\Lambda_n^c}$ for $y \in (\Lambda_n \times S)^k$. We would like to fix the bond set B before we choose the particle states y_i. Thus we introduce bonds between indices of particles instead of bonds between particles. Let $\mathbb{N}_k := \{1, \ldots, k\}$, $\bar{Y}^k := \mathbb{N}_k \cup \bar{Y}_{\Lambda_n^c}$, and $E_n(\bar{Y}^k) := \{y_1y_2 \in E(\bar{Y}^k) : y_1y_2 \cap \mathbb{N}_k \neq \emptyset\}$. For $B \subset E_n(\bar{Y}^k)$ and $y \in (\Lambda_n \times S)^I$ ($I \subset \mathbb{N}_k$) we define B_y to be the bond set constructed from B by replacing the point $i \in I$ by y_i in every bond of B and by deleting every bond B that contains a point $i \in \mathbb{N}_k \setminus I$. Analogously let $\bar{Y}_y := \{y_i : i \in I\} \cup \bar{Y}_{\Lambda_n^c}$ be the configuration corresponding to the sequence and let $(\bar{Y}, B)_y := (\bar{Y}_y, B_y)$. We obtain

$$I(k) = \sum_{B \subset E_n(\bar{Y}^k)}{}' I(k,B), \quad \text{where } I(k,B) := \int_{{\Lambda_n}^k} dy\, (f \circ \mathfrak{T}_{R,n} \cdot \varphi_{R,n})(\bar{Y}, B)_y.$$

To compute $I(k,B)$ we need to calculate $\mathfrak{T}_{R,n}(\bar{Y}, B)_y$, and for this we must identify the points of $P_i^{\bar{Y}_y, B_y}$ among $\bar{Y}_y$. So let Π_k the set of all sequences $\eta = (\eta_j)_{0 \leq j \leq m}$ of disjoint nonempty subsets of $\bar{Y}^k$ such that $\bar{Y}_{\Lambda_n^c} \subset \eta_0$ and every B-cluster of $(\bar{Y}^k, B)$ has nonempty intersection with exactly one of the sets η_j, i.e. the B-clusters η_j^B of the sets η_j define a partition of $\bar{Y}^k$. Let the length of the sequence be denoted by $m(\eta) := m$. For $\eta \in \Pi_k$ and $y \in (\Lambda_n \times S)^k$ let $\eta_y = (\eta_{j,y})_{0 \leq j \leq m(\eta)}$ and $\eta_y^B = (\eta_{j,y}^B)_{0 \leq j \leq m(\eta)}$ be the sequences corresponding

to η and η^B, where every i is replaced by y_i. For $\eta \in \Pi_k$ let

$$A_{k,B,\eta} := \{y \in (\Lambda_n \times S)^k : m(\eta) = m(\bar{Y}_y, B_y),\, \forall\, j \geq 0 : \eta_{j,y} = P_j^{\bar{Y}_y,B_y}\},$$
$$\tilde{A}_{k,B,\eta} := \{y \in (\Lambda_n \times S)^k : m(\eta) = \tilde{m}(\bar{Y}_y, B_y),\, \forall\, j \geq 0 : \eta_{j,y} = \tilde{P}_j^{\bar{Y}_y,B_y}\},$$

where $\tilde{m}(\bar{Y}_y, B_y)$ and $\tilde{P}_j^{\bar{Y}_y,B_y}$ are the objects from the construction of the inverse transformation in Subsection 6.5. We note that

$$\forall\, y \in A_{k,B,\eta} : \eta^B_{j,y} = C_j^{\bar{Y}_y,B_y} \quad \text{ and } \quad \forall\, y \in \tilde{A}_{k,B,\eta} : \eta^B_{j,y} = \tilde{C}_j^{\bar{Y}_y,B_y}.$$

Now we can write

$$I(k,B) \;=\; \sum_{\eta\in\Pi_k} \int dy\, 1_{A_{k,B,\eta}}(y)(f \circ \mathfrak{T}_{R,n} \cdot \varphi_{R,n})(\bar{Y}, B)_y,$$

and we denote the summands in the last term by $I(k,B,\eta)$. If $y \in A_{k,B,\eta}$ we can derive a simple expression for $\mathfrak{T}_{R,n}(\bar{Y}, B)_y$: For $i \in \eta^B_j$ the translation distance of y_i doesn't depend on all components of y, but only on those y_l such that $l \in \eta^B_{j'}$ for some $j' \leq j-1$ and additionally on those y_l such that $l \in \eta_j$ whenever $i \notin \eta_j$. Hence for $y \in (\Lambda_n \times S)^k$, $\eta \in \Pi_k$ and $0 \leq j \leq m(\eta)$ we define $y^{\eta,j}$ to be the subsequence of y corresponding to the index set $\bigcup_{j'\leq j} \eta^B_{j'}$, we define a formal translation distance and a formal transformation by

$$\begin{aligned}
&t_j^{B,\eta,y} := t_j^{(\bar{Y},B)_{y^{\eta,j-1}}} \qquad \text{and} \qquad T^{B,\eta}(y) := (T^{B,\eta,y}_{j(i)}(y_i))_{1\leq i\leq k}, \qquad \text{where}\\
&j(i) := j \ \text{ for } \ i \in \eta_j \qquad \text{and} \qquad j(i) := (j,*) \ \text{ for } \ i \in \eta^B_j \setminus \eta_j,\\
&T_j^{B,\eta,y} := id + t_j^{B,\eta,y}\, \vec{e} \ \text{ and } \ T_{j,*}^{B,\eta,y} := id + t_j^{B,\eta,y}(y_{\min \eta_j})\vec{e}.
\end{aligned}$$

Then

$$y \in A_{k,B,\eta} \;\Rightarrow\; \begin{cases} \mathfrak{T}_n(\bar{Y}, B)_y \;=\; (\bar{Y}, B)_{T^{B,\eta}(y)} & \text{and} \\ T_j^{\bar{Y}_y,B_y} \;=\; T_j^{B,\eta,y} \ \text{ for all } 0 \leq j \leq m(\eta) \end{cases} \tag{6.20}$$

holds by definition. Furthermore we observe that for all $y \in (\mathbb{R}^2_S)^k$ we have

$$y \in A_{k,B,\eta} \quad \Leftrightarrow \quad T^{B,\eta}(y) \in \tilde{A}_{k,B,\eta}. \tag{6.21}$$

Here "$\Rightarrow$" holds by (6.20) and (6.19) from the proof of Lemma 25. For "$\Leftarrow$" let $y \in (\mathbb{R}^2_S)^k$ such that $T^{B,\eta}(y) \in \tilde{A}_{k,B,\eta}$ and let $(Y', B') := \tilde{\mathfrak{T}}_{R,n}(\bar{Y}, B)_{T^{B,\eta}(y)}$, where $\tilde{\mathfrak{T}}_{R,n}$ is the inverse of $\mathfrak{T}_{R,n}$ as defined in the last subsection. By induction on j we can show

$$\forall\, 0 \leq j \leq m(\eta): \quad T_j^{Y',B'} = T_j^{B,\eta,y},\ \eta_{j,y} = P_j^{Y',B'}, \ \text{ and } \ \eta^B_{j,y} = C_j^{Y',B'}.$$

In the inductive step $j-1 \to j$ the first assertion follows from the induction hypothesis, the third follows from the second, and the second follows from the first, the bijectivity of $T_j^{Y',B'}$, and $T_j^{B,\eta,y}(\eta_{j,y}) = T_j^{Y',B'}(P_j^{Y',B'})$, which holds as

$$T_j^{B,\eta,y}(\eta_{j,y}) \;=\; \eta_{j,T^{B,\eta}(y)} \;=\; \tilde{P}_j^{(\bar{Y},B)_{T^{B,\eta}(y)}} \;=\; P_j^{Y',B'} + \tau_j^{Y',B'} \;=\; T_j^{Y',B'}(P_j^{Y',B'}),$$

where we have used the definition of $\tilde{A}_{k,B,\eta}$ and (6.19) from the proof of Lemma 25. This completes the proof of the above assertion and we conclude that $(\bar{Y}_y, B_y) = (Y', B')$, which implies $\eta_{j,y} = P_j^{Y',B'} = P_j^{\bar{Y}_y,B_y}$ and thus (6.21). Defining $g(y) := 1_{\tilde{A}_{k,B,\eta}}(y) f(\bar{Y}_y, B_y)$, (6.20) and (6.21) imply

$$I(k,B,\eta) \;=\; \Big[\prod_{j=0}^{m(\eta)} \Big(\prod_{i\in\eta_j^B\cap\mathbb{N}_k} \int dy_i \Big)\Big(\prod_{i'\in\eta_j\cap\mathbb{N}_k} \big|1+\partial_{\vec{e}} t_j^{B,\eta,y}(y_{i'})\big| \Big)\Big]\, g(T^{B,\eta}(y)),$$

where we have also inserted the definition of $\varphi_{R,n}$ (5.11). Now we transform the integrals. For $j = m(\eta)$ to 0 and $i \in \eta_j^B \cap \mathbb{N}_k$ we substitute $y_i' := T_{j(i)}^{B,\eta,y} y_i$. For $i \in \eta_j^B \setminus \eta_j$ $T_{j(i)}^{B,\eta,y}$ is a translation by a constant vector, so $dy_i' = dy_i$. For $i \in \eta_j$ the transformation only concerns the first spatial component of y_i, and Lemma 19 implies $dy_i' = \big|1 + \partial_{\vec{e}} t_j^{B,\eta,y}(y_i)\big| dy_i$. So

$$I(k,B,\eta) \;=\; \Big[\prod_{j=0}^{m(\eta)} \Big(\prod_{i\in\eta_j^B\cap\mathbb{N}_k} \int dy_i' \Big)\Big]\, g(y') \;=\; \int dy\, 1_{\tilde{A}_{k,B,\eta}}(y) f(\bar{Y}_y, B_y),$$

and we are done as the same arguments show that the right hand side of (5.12) equals

$$e^{-4n^2} \sum_{k\geq 0} \frac{1}{k!} \sideset{}{'}\sum_{B\subset E_n(\bar{Y}^k)} \sum_{\eta\in\Pi_k} \int dy\, 1_{\tilde{A}_{k,B,\eta}}(y) f(\bar{Y}_y, B_y).$$

Analogously the density function can be shown to be well defined: For all $\bar{Y} \in \mathcal{Y}$

$$\nu_{\Lambda_n} \otimes \pi_n'(\text{“}\varphi_{R,n} \text{ well defined''}|\bar{Y}) \;=\; e^{-4n^2} \sum_{k\geq 0} \frac{1}{k!} \sideset{}{'}\sum_{B\subset E_n(\bar{Y}^k)} \sum_{\eta\in\Pi_k} I'(k,B,\eta), \text{ with}$$

$$I'(k,B,\eta) := \Big[\prod_{j=0}^{m(\eta)} \Big(\prod_{i\in\eta_j^B\cap\mathbb{N}_k} \int dy_i \Big)\Big(\prod_{i'\in\eta_j\cap\mathbb{N}_k} 1_{\{\partial_{\vec{e}} t_j^{B,\eta,y}(y_{i'}) \text{ exists}\}} \Big)\Big] 1_{A_{k,B,\eta}}(y).$$

As $t_j^{B,\eta,y}$ is $\vec{e}$-hLd, we have for arbitrary $r \in \mathbb{R}$, $\sigma \in S$, k, B, η, and y as above that $\partial_{\vec{e}} t_j^{B,\eta,y}(.,r,\sigma)$ exists a.s.. So we may replace all indicator functions in the above product by 1 using Fubini's theorem, and the above probability equals 1.

6.7 Key estimates: Lemma 14

For (5.14) let $(Y,B) \in G_{R,n}$. By Lemma 19 we have $|\partial_{\vec{e}} t_k^{Y,B}(.)| \leq 1/2$. Using the inequality $-\log(1-a) \leq 2a$, which holds for $0 \leq a \leq 1/2$, we thus obtain

$$\begin{aligned} f_{R,n}(Y,B) &:= -\log\varphi_{R,n}(Y,B) - \log\bar{\varphi}_{R,n}(Y,B) \\ &= -\sum_{k=0}^{m(Y,B)} \sum_{y\in P_k^{Y,B}} \log\big(1 - (\partial_{\vec{e}} t_k^{Y,B}(y))^2\big) \;\leq\; \sum_{k=0}^{m(Y,B)} \sum_{y\in P_k^{Y,B}} 2(\partial_{\vec{e}} t_k^{Y,B}(y))^2. \end{aligned}$$

By Lemma 19 $\partial_{\vec{e}} t_k^{Y,B}(y)$ equals either 0 or $\partial_{\vec{e}} t_0^{Y,B}(y)$ or $\partial_{\vec{e}} m_{y',t^{Y,B}(y')}(y)$ for some $y' \in Y$ with $(y', y) \in K''$. Using (6.7) we see that

$$\|\partial_{\vec{e}} m_{y',t^{Y,B}(y')}\| \le (\tau_{R,n}(|y'| - c_K) - t^{Y,B}(y'))c_f,$$
$$\text{where} \quad t^{Y,B}(y') \ \ge\ \tau_{R,n}^{\wedge,Y,B_+}(y') \ =\ \bigwedge_{y'' \in C_{Y,B_+}} \tau_{R,n}(|y''|).$$

Furthermore for $y = (r_1, r_2, \sigma) \in \mathbb{R}_S^2$ we have

$$|\partial_{\vec{e}} t_0^{Y,B}(y)| \ =\ 1_{\{n \ge |r_1| > |r_2| \vee R\}} \tau \frac{q(|r_1| - R)}{Q(n-R)} \ \le\ 1_{\{y \in \Lambda_n\}} \tau \frac{q(|y| - R)}{Q(n-R)},$$

so we can estimate $f_{R,n}(Y, B)$ by the sum of the following two expressions:

$$\begin{aligned} \Sigma_2(R,n,Y) \ &:=\ 2\tau^2 \sum_{y \in Y} 1_{\{y \in \Lambda_n\}} \frac{q(|y| - R)^2}{Q(n-R)^2}, \\ \Sigma_3(R,n,Y,B) \ &:=\ 2c_f^2 \sum_{y,y',y'' \in Y} 1_{K''}(y', y) 1_{\{y' \overset{Y,B_+}{\longleftrightarrow} y''\}} \tau_{R,n}^q(y', y''), \end{aligned} \tag{6.22}$$

where we have used the shorthand notation (6.9). Using these terms in the definition (5.5) of $G_{R,n}$ we get (5.14). For a proof of (5.15) we first note that for all $y, y' \in \mathbb{R}_S^2$, $\vartheta \in [-1, 1]$ with $(y, y' + s\vec{e}) \notin K$ for all $s \in [-\vartheta, \vartheta]$ we can estimate $\bar{U}(y, y' + \vartheta\vec{e}) + \bar{U}(y, y' - \vartheta\vec{e}) - 2\bar{U}(y, y')$ by

$$\varphi_{y,y'}^{\bar{U}}(\vartheta) + \varphi_{y,y'}^{\bar{U}}(-\vartheta) - 2\varphi_{y,y'}^{\bar{U}}(0) \ \le\ \sup_{s \in [-\vartheta, \vartheta]} \frac{d^2}{dt^2} \varphi_{y,y'}^{\bar{U}}(s)\vartheta^2 \ \le\ \psi(y, y')\vartheta^2,$$

using Taylor expansion of $\varphi_{y,y'}^{\bar{U}}$ and the ψ-domination of the $\vec{e}$-derivatives. Now let $(Y, B) \in G_{R,n}$. W.l.o.g. we may assume that the right hand side of (5.15) is finite. Introducing $\vartheta_{y,y'} := t^{Y,B}(y') - t^{Y,B}(y)$ for $y, y' \in E_n(Y)$ we have

$$\begin{aligned} &H_{\Lambda_n}^{\bar{U}}(\mathfrak{T}_{R,n}^B Y) \ +\ H_{\Lambda_n}^{\bar{U}}(\bar{\mathfrak{T}}_{R,n}^B Y) \ -\ 2H_{\Lambda_n}^{\bar{U}}(Y) \\ &= \sum_{yy' \in E_n(Y)} [\bar{U}(y, y' + \vartheta_{y,y'}\vec{e}) + \bar{U}(y, y' - \vartheta_{y,y'}\vec{e}) - 2\bar{U}(y, y')] \\ &\le \sum_{yy' \in E_n(Y)} \psi(y, y')\,(t^{Y,B}(y) - t^{Y,B}(y'))^2 \quad =: \quad f_{R,n}(Y, B). \end{aligned}$$

In the first step we have used that $\bar{U}$ is $\vec{e}$-invariant, and in the second step we are allowed to apply the above Taylor estimate as for $(y, y') \notin K$ we have $(y, y' + s\vec{e}) \notin K$ for all $s \in [-\vartheta_{y,y'}, \vartheta_{y,y'}]$ by (6.4), and for $(y, y') \in K$ we have $\vartheta_{y,y'} = 0$ by (5.8). Applying the arithmetic-quadratic mean inequality to

$$\Big((t^{Y,B}(y) - \tau_{R,n}(|y|)) + (\tau_{R,n}(|y|) - \tau_{R,n}(|y'|)) + (\tau_{R,n}(|y'|) - t^{Y,B}(y'))\Big)^2$$

we obtain

$$\begin{aligned} f_{R,n}(Y, B) \ \le\ & 6 {\sum_{y,y' \in Y}}^{\neq} \psi(y, y')\,(\tau_{R,n}(|y|) - t^{Y,B}(y))^2 \\ &+ 3 {\sum_{y,y' \in Y}}^{\neq} 1_{\{|y| \le |y'|\}}\, \psi(y, y')\,(\tau_{R,n}(|y|) - \tau_{R,n}(|y'|))^2. \end{aligned}$$

In the first sum on the right hand side we again use (6.7) to estimate

$$(\tau_{R,n}(|y|) - t^{Y,B}(y))^2 \le \sum_{y''\in Y} 1_{\{|y|\le|y''|\}} 1_{\{y \overset{Y,B_+}{\longleftrightarrow} y''\}} (\tau_{R,n}(|y|) - \tau_{R,n}(|y''|))^2,$$

so $f_{R,n}(Y,B)$ can be estimated by the sum of the two following expressions:

$$\begin{aligned} \Sigma_4(R,n,Y) &:= 3 \sum_{y,y'\in Y}^{\neq} \psi(y,y')\tau^q_{R,n}(y,y'), \\ \Sigma_5(R,n,Y,B) &:= 6 \sum_{y,y',y''\in Y} 1_{\{y \overset{Y,B_+}{\longleftrightarrow} y''\}} \psi(y,y')\tau^q_{R,n}(y,y''). \end{aligned} \tag{6.23}$$

Inserting these sums into the definition of $G_{R,n}$ in (5.5), we obtain (5.15).

6.8 Set of good configurations: Lemma 15

The set of good configurations $G_{R,n}$ is defined in terms of the cluster range $r^{Y,B_+}_{n'}$ and the functions $\Sigma_i(R,n,Y,B)$, see (5.5). We will show that the $\mu\otimes\pi_n$-expectation of $r^{Y,B_+}_{n'}$ is finite and independent of n and that for fixed R the expectation of every $\Sigma_i(R,n,Y,B)$ tends to 0 for $n\to\infty$. Then Markov's inequality implies the desired result: We can first choose $R > n'$ such that $\mu\otimes\pi_n(r^{Y,B_+}_{n'} \ge R) < \delta/2$ for all n, and we may then choose an $n > R$ such that $\mu\otimes\pi_n(\sum_{i=1}^5 \Sigma_i(R,n,Y,B) \ge \delta) < \delta/2$.

For $Y\in\mathcal{Y}$, $B\subset E(Y)$ and any path $y_0,...,y_m$ in the graph (Y,B_+) such that $y_0\in\Lambda_{n'}$ we have $|y_m| \le n' + \sum_{k=1}^m |y_k - y_{k-1}|$. By considering all possibilities for such paths we thus obtain

$$r^{Y,B_+}_{n'} \le n' + \sum_{m\ge1} \sum_{y_0,\dots,y_m\in Y}^{\neq} 1_{\{y_0\in\Lambda_{n'}\}} \prod_{i=1}^m 1_{\{y_i y_{i-1}\in B_+\}} \sum_{k=1}^m |y_k - y_{k-1}|.$$

Under the Bernoulli measure $\pi_n(dB|Y)$ the events $\{y_i y_{i-1}\in B_+\}$ are independent, and for $g:(\mathbb{R}^2_S)^2\to\mathbb{R}$, $g := 1_{K''\setminus K^U} + \tilde u$ we have

$$\int \pi_n(dB|Y) 1_{\{y_i y_{i-1}\in B_+\}} \le 1_{K^U}(y_{i-1},y_i) + g(y_{i-1},y_i). \tag{6.24}$$

Using the hard core property (3.2) and Lemma 6 we obtain

$$\begin{aligned} r &:= \int \mu(dY) \int \pi_n(dB|Y) r^{Y,B_+}_{n'} - n' \\ &\le \sum_{m\ge1}\sum_{k=1}^m \int \mu(dY) \sum_{y_0,\dots,y_m\in Y}^{\neq} 1_{\{y_0\in\Lambda_{n'}\}} |y_k - y_{k-1}| \prod_{i=1}^m g(y_{i-1},y_i) \\ &\le \sum_{m\ge1}\sum_{k=1}^m (z\xi)^{m+1} \int dy_0\dots dy_m 1_{\{y_0\in\Lambda_{n'}\}} |y_k - y_{k-1}| \prod_{i=1}^m g(y_{i-1},y_i). \end{aligned}$$

Setting $c_g := (1+c_K^2)c_\xi + c_u$, we conclude from (5.1) and (5.2) that we have

$$\begin{aligned}
&\int g(y,y')\,dy' \;\le\; c_\xi \qquad \text{and} \\
&\int g(y,y')|y-y'|\,dy' \;\le\; \int g(y,y')(1+|y-y'|^2)\,dy' \;\le\; c_g
\end{aligned} \tag{6.25}$$

for all $y \in \mathbb{R}^2_S$, hence we can estimate the integrals over dy_i in the above expression beginning with $i=m$. These estimates give $m-1$ times a factor c_ξ and one time a factor c_g. Finally the integration over dy_0 gives an additional factor $\lambda^2(\Lambda_{n'}) = (2n')^2$. Thus

$$r \;\le\; (2n'z\xi)^2 c_g \sum_{m\ge 1} m(c_\xi z\xi)^{m-1} \;<\; \infty, \qquad \text{as } c_\xi z\xi < 1.$$

This gives the finiteness of the expectation of the cluster range. The functions $\Sigma_i(R,n,Y,B)$ have been specified in (6.8), (6.22) and (6.23):

$$\begin{aligned}
&\Sigma_1 = 4c_f^2 \sum_{y,y'\in Y} 1_{\{y \overset{Y,B_+}{\longleftrightarrow} y'\}} \tau^q_{R,n}(y,y'), \quad \Sigma_4 = 3\sum_{y,y'\in Y}^{\neq} \psi(y,y')\tau^q_{R,n}(y,y'),\\
&\Sigma_2 = 2\tau^2 \sum_{y\in Y} 1_{\{y\in\Lambda_n\}} \frac{q(|y|-R)^2}{Q(n-R)^2}, \quad \Sigma_5 = 6\sum_{y,y',y''\in Y} 1_{\{y \overset{Y,B_+}{\longleftrightarrow} y'\}} \psi(y,y'')\tau^q_{R,n}(y,y'),\\
&\Sigma_3 = 2c_f^2 \sum_{y,y',y''\in Y} 1_{\{y \overset{Y,B_+}{\longleftrightarrow} y'\}} 1_{K''}(y,y'')\tau^q_{R,n}(y,y').
\end{aligned}$$

We start with an estimate on $\tau^q_{R,n}$. For $s' > s$ such that $s' > R$ and $s < n$,

$$0 \;\le\; r(s-R,n-R) - r(s'-R,n-R) \;=\; \int_{R\vee s}^{s'\wedge n} \frac{q(t-R)}{Q(n-R)}dt \;\le\; (s'-s)\,\frac{q(s-R)}{Q(n-R)}$$

by the monotonicity of q. Defining $\bar n := n + c_K$ and $\bar R := R + c_K$ we thus have

$$\tau^q_{R,n}(y,y') \;\le\; 1_{\{y\in\Lambda_{\bar n}\}}\tau^2\,(|y'|-|y|+c_K)^2\,\frac{q(|y|-\bar R)^2}{Q(\bar n-\bar R)^2} \qquad \text{for } y,y' \in \mathbb{R}^2_S, \tag{6.26}$$

using the substitution $s' := |y'|$ and $s := |y| - c_K$. (If $s' \le R$ or $s \ge n$ then $\tau^q_{R,n}(y,y') = 0$.) The following relations will give us control over the relevant terms of the right hand side of (6.26). For $\bar n \ge 2\bar R$ we have

$$\begin{aligned}
\int_{\Lambda_{\bar n}} dy\, q(|y|-\bar R)^2 \quad &\le \quad \int_0^{2\bar R} ds\, 8s \;+\; \int_{\bar R}^{\bar n-\bar R} ds\, 8(s+\bar R)q(s)^2 \\
&\le\; 16\bar R^2 \;+\; 32\int_0^{\bar n-\bar R} q(s)ds \quad \le \quad 16\bar R^2 \;+\; 32 Q(\bar n-\bar R).
\end{aligned}$$

In the first step we used $q \le 1$, and in the second step $\bar R \le s$ and $sq(s) \le 2$. As $\lim_{n\to\infty} Q(n) = \infty$ by $\log\log n \le Q(n)$ for $n > 1$, the above implies

$$\lim_{n\to\infty} c(R,n) \;=\; 0 \qquad \text{for} \qquad c(R,n) \;:=\; \int_{\Lambda_{\bar n}} dy\,\frac{q(|y|-\bar R)^2}{Q(\bar n-\bar R)^2}. \tag{6.27}$$

Finally, for $y_0, \dots, y_m \in \mathbb{R}^2_S$ we deduce from the triangle inequality that

$$\Big||y_m|-|y_0|+c_K\Big| \le m \bigvee_{k=1}^m |y_k-y_{k-1}|+c_K \le (m+1)(1\vee c_K)\Big(1\vee \bigvee_{k=1}^m |y_k-y_{k-1}|\Big),$$

$$\text{so} \quad (|y_m|-|y_0|+c_K)^2 \le (m+1)^2(1\vee c_K^2) \bigvee_{k=1}^m (1\vee |y_k-y_{k-1}|^2). \tag{6.28}$$

No we will proceed as in the first part of this section: For $Y \in \mathcal{Y}$ and $B \subset E(Y)$ we can estimate the summands of $\Sigma_1(R,n,Y,B)$ by considering all paths $y_0, \dots, y_m$ in the graph (Y, B_+) connecting $y = y_0$ and $y' = y_m$. By (6.26) and (6.28) we can estimate $\Sigma_1(R,n,Y,B)$ by a constant c times

$$\sum_{m\ge 0}(m+1)^2\sum_{k=1}^m \sum^{\neq}_{y_0,\dots,y_m\in Y} 1_{\{y_0\in\Lambda_{\bar n}\}}\frac{q(|y_0|-\bar R)^2}{Q(\bar n-\bar R)^2}(1\vee|y_k-y_{k-1}|^2)\prod_{i=1}^m 1_{\{y_iy_{i-1}\in B_+\}}.$$

The expectation of the last term can be estimated using (6.24), Lemma 6 and (6.24), and thus we get the following upper bound for the expectation of Σ_1:

$$(z\xi)^2 c_g c\sum_{m\ge 0}(m+1)^3(c_\xi z\xi)^{m-1}c(R,n).$$

Similarly we estimate the summands of $\Sigma_5(R,n,Y,B)$ by considering all paths $y_0, \dots, y_m$ in the graph (Y, B_+) connecting $y = y_0$ and $y' = y_m$ and by distinguishing the cases $y_j = y''$ and $y_j \ne y''\,\forall\, j$. By (6.26) and (6.28) we can estimate $\Sigma_5(R,n,Y,B)$ by a constant c times

$$\begin{aligned}\sum_{m\ge 0}(m+1)^2\sum_{k=1}^m \sum^{\neq}_{y_0,\dots,y_m\in Y} 1_{\{y_0\in\Lambda_{\bar n}\}}\frac{q(|y_0|-\bar R)^2}{Q(\bar n-\bar R)^2}(1\vee|y_k-y_{k-1}|^2)\\ \times\prod_{i=1}^m 1_{\{y_iy_{i-1}\in B_+\}}\Big[\sum_{y''\in Y, y''\ne y_j\,\forall\, j}\psi(y_0,y'')+\sum_{j=0}^m\psi(y_0,y_j)\Big].\end{aligned}$$

We estimate the second sum in the brackets by $c_\psi(m+1)$ using (5.2). Proceeding as above we see that the expectation of $\Sigma_5(R,n,Y,B)$ can be estimated by

$$(z\xi)^2 c_g c\sum_{m\ge 0}(m+1)^3(c_\xi z\xi)^{m-1}c(R,n)\Big(z\xi c_\psi + c_\psi(m+1)\Big).$$

We estimate $\Sigma_3(R,n,Y,B)$ by a similar term as the one for Σ_5, where the function ψ is now replaced by $1_{K''}$, so the expectation of $\Sigma_3(R,n,Y,B)$ can be estimated by

$$(z\xi)^2 c_g c\sum_{m\ge 0}(m+1)^3(c_\xi z\xi)^{m-1}\,c(R,n)(z\xi c_\xi + m+1).$$

Analogously we can estimate the expectation of $\Sigma_4(R,n,Y,B)$ by

$$c(z\xi)^2\int_{\Lambda_{\bar n}} dy\,\frac{q(|y|-\bar R)^2}{Q(\bar n-\bar R)^2}\int dy'\psi(y,y')(1\vee|y-y'|^2) \le c(z\xi)^2 c(R,n)c_\psi$$

for some constant c, and finally the expectation of $\Sigma_2(R, n, Y)$ can be estimated by $2z\xi\tau^2 c(R, n)$. In the bounds of the expectations of the above terms the sums over m have finite values by (5.1), so we are done by (6.27).

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